Rational Functions:

Rules of derivatives: \((u\text{ and } v\text{ are functions of } x\text{ and } a, b\text{ and } k\text{ are real numbers})\)

1) \((uv)' = u'v + v'u\)
2) \(\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}\)
3) \((u \pm v)' = u' \pm v'\)

4) \(\left(\frac{k}{v}\right)' = \frac{-kv'}{v^2}\)
5) \((kuv)' = kvu'\)
6) \((u^n)' = n u^{n-1}u'\)

7) \((k \pm u)' = u'\)
8) \((ax^n)' = nax^{n-1}\)
9) \((ax^2 + bx + c)' = 2ax + b\)
10) \((ax + b)' = a\)
11) \(k' = 0\)
12) \((u^2)' = 2uu'\)
13) \((\tan x)' = \frac{1}{(\cos x)^2}\)
14) \((\cot x)' = -\frac{1}{(\sin x)^2}\)
15) \((\sqrt{u})' = \frac{u'}{2\sqrt{u}}\)

Rules of limits:

A) Determinate cases:

1) \((+\infty)(+\infty) = +\infty\)
2) \((+\infty)(-\infty) = -\infty\)
3) \((-\infty)(-\infty) = +\infty\)
4) \((+\infty) + (+\infty) = +\infty\)
5) \(-\infty - \infty = -\infty\)
6) \(\frac{1}{0^+} = +\infty\)
7) \(\frac{1}{0^-} = 0^+\)
8) \(\frac{1}{0^-} = 0^-\)
9) \(\frac{0}{0^+} = 0\)
10) \(\frac{0}{0^-} = \infty\)

13) \(k + \infty = +\infty\)
14) \(k - \infty = -\infty\)
15) \((k)(+\infty) = \begin{cases} +\infty \text{ if } k > 0 \\ -\infty \text{ if } k < 0 \end{cases}\)
16) \(\frac{+\infty}{k} = \begin{cases} +\infty \text{ if } k > 0 \\ -\infty \text{ if } k < 0 \end{cases}\)
17) \(\frac{-\infty}{k} = \begin{cases} +\infty \text{ if } k > 0 \\ -\infty \text{ if } k < 0 \end{cases}\)
18) \(\frac{+\infty}{0^+} = +\infty\)
19) \(\frac{-\infty}{0^-} = -\infty\)

B) Indeterminate cases:

1) \(\infty \cdot \infty\)
2) \(\infty - \infty\)
3) \(0(\infty)\)
4) \(\frac{0}{0}\)

Limit of a polynomial function at infinity:

\[
\lim_{x \to \infty} (a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n) = \lim_{x \to \infty} a_nx^n
\]

Limit of a rational function at infinity:

\[
\lim_{x \to \infty} \frac{a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_{m-1}x^{m-1} + b_mx^m} = \lim_{x \to \infty} \frac{a_nx^n}{b_mx^m}
\]

where \(a_n \neq 0\) and \(b_m \neq 0\)

Asymptotes:

lim \(f(x) = \pm\infty\) then \(x = a\) is vertical asymptote to curve (C) of \(f\).

lim \(f(x) = b\) then \(y = b\) is horizontal asymptote to curve (C) of \(f\).

lim \([f(x) - ax - b] = 0\) then \(y = ax + b\) is an oblique asymptote to curve (C) of \(f\).
**Sense of variation of a numerical function $f$:**

Let $f$ be a numerical function differentiable function over an interval $I$ and $f'$ its derivative.

1. $f$ is increasing over $I$ if $f'(x) \geq 0$ for every real number $x$ of $I$.
2. $f$ is strictly increasing over $I$ if $f'(x) > 0$ for every real number $x$ of $I$.
3. $f$ is decreasing over $I$ if $f'(x) \leq 0$ for every real number $x$ of $I$.
4. $f$ is strictly decreasing over $I$ if $f'(x) < 0$ for every real number $x$ of $I$.
5. $f$ is constant over $I$ if $f'(x) = 0$ for every real number $x$ of $I$.

**Center of symmetry:**

Let $f$ be a function with domain $I$ and $(C)$ its representative curve in an orthonormal system $(O, \vec{i}, \vec{j})$.

$(C)$ admits the point $I(a, b)$ as center of symmetry iff:

1. $a$ is the center of domain $I$.
2. $f(2a - x) + f(x) = 2b$ or $f(a - x) + f(a + x) = 2b$ for all $x$ belong to $I$.

**Exercise 1:**

Let $f$ be the function defined over $\mathbb{R}$ by $f(x) = \frac{x^2 + x + 2}{x - 1}$.

Designate by $(C)$ the representative curve of $f$ in an orthonormal system $(O, \vec{i}, \vec{j})$.

1. a) Determine the real numbers $a$, $b$, and $c$ such that: $f(x) = ax + b + \frac{c}{2x + 3}$
   b) Show that $y = ax + b$ is an oblique asymptote to $(C)$.
   c) Determine limit of $f$ at $\frac{-3}{2}$ and deduce an asymptote to $(C)$.

2) Calculate the derivative of $f$ then draw its variation.
3) Trace $(C)$.

**Exercise 2:**

Consider the function $f$ defined over $\mathbb{R}$ by $f(x) = \frac{x^2 + x + 2}{x - 1}$.

Designate by $(C)$ its representative curve in an orthonormal system $(O, \vec{i}, \vec{j})$.

1) Determine limit of $f$ as $x$ tends to 1 and interpret the result graphically.
2) Show that $f(x) = x + 2 + \frac{4}{x - 1}$.
3) Show that the point $A(1,3)$ is a center of symmetry of $(C)$ over $\mathbb{R}\setminus\{1\}$.
4) Study the variations of $f$ and trace $(C)$. 
Exercise 3:

Consider the function $f$ defined over $]-\infty; -2[ \cup ] -2; +\infty[$ by $f(x) = ax + b + \frac{2}{x + 2}$ where $a$ and $b$ are two real numbers ($a \neq 0$). Denote by $(C)$ its representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$.

The table of variations of $f$ is the following:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\infty$</th>
<th>$-2$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$-\infty$</td>
<td>$+\infty$</td>
<td>$-\infty$</td>
</tr>
</tbody>
</table>

A) 1) Compare, with justification, $f(-4)$ and $f(-3)$.

2) What is the number of solutions of the equation $f(x) = -2$?

3) Knowing that $f(-3) = 0$ and that $f(0) = 0$, calculate $a$ and $b$.

B) In this part, suppose that $f(x) = -x - 1 + \frac{2}{x + 2}$.

1) Verify that the line $(\Delta)$ with equation $x = -2$ and the line $(D)$ with equation $y = -x - 1$ are asymptotes to $(C)$.

2) Draw $(\Delta)$, $(D)$ and $(C)$.

3) a- The line $(d)$ with equation $y = x$ intersects the curve $(C)$ in two distinct points. Calculate the coordinates of these points and draw $(d)$ in the same system $(O; \vec{i}, \vec{j})$.

b- Using $(C)$ and $(d)$, solve the following inequality:

$$-x - 1 + \frac{2}{x + 2} \geq x.$$
Exercise 4:

The curve (C) shown below is the representative curve, in an orthonormal system, of a function $f$ that is defined on $]-\infty ; 2[ \cup ] 2 ; +\infty [.$

The lines (d) and (D) are asymptotes of (C).

1) Determine $\lim_{x \to +\infty} f(x)$

and $\lim_{x \to -\infty} f(x)$.

2) Find the equations of the asymptotes of (C).

3) Solve each of the following three equations:

   - $f(x) = 1$
   - $f(x) = 5$
   - $f'(x) = 0$.

4) Solve each of the following two inequalities:

   - $f(x) > 3$
   - $f'(x) \leq 0$.

5) Set up the table of variations of $f$.

6) Knowing that $f(x) = ax + b + \frac{1}{x + c}$

   show that $a = b = 1$ and $c = -2$.

8) Prove that $I(2 ; 3)$ is a center of symmetry of (C).
Distributions in two variables:

Distribution in two variables:

It is a set defined on the same population having two variables $X$ and $Y$ such that a pair of values is defined for every member of the population.

Positive relationship:

A relation between two variables is positive if the two variables vary in the same sense.

Negative relationship:

A relation between two variables is negative if the two variables vary in opposite sense.

Mean point:

The point $G(\bar{X}; \bar{Y})$ is called the center of gravity or the mean point of the series $(X,Y)$ where:

\[
\bar{X} = \frac{1}{n} \sum_{i} X_i \quad \text{and} \quad \bar{Y} = \frac{1}{n} \sum_{i} Y_i
\]

Variance:

\[
V_X = \frac{1}{n} \sum_{i} X_i^2 - \bar{X}^2 \quad \text{and} \quad V_Y = \frac{1}{n} \sum_{i} Y_i^2 - \bar{Y}^2
\]

Standard deviation:

\[
\sigma_X = \sqrt{V(X)} \quad \text{and} \quad \sigma_Y = \sqrt{V(Y)}
\]

Covariance:

\[
Cov(X; Y) = \frac{1}{n} \sum_{i} X_i Y_i - \bar{X} \bar{Y}
\]

Linear adjustment:

The procedure consists of adjusting a straight line through the data points in such a way that it is as near as possible to the data points.

Method of least squares:

This method consists of adjusting a line through the data points in such a way that the sum of the squares of the distance between the data points and the line is minimal.
**Regression line:**

In the case where a linear adjustment is justified, we determine the equation of the regression line \((D_{y,x})\) of \(y\) in \(x\) by using the least square method: \(y = ax + b\)

with \(a = \frac{\text{cov}(X;Y)}{V_x}\) and \(b = \bar{Y} - a\bar{X}\)

There exists another regression line \((D'_{x,y})\) in \(y\), of equation: \(x = a' y + b'\) where \(a' = \frac{\text{cov}(X;Y)}{V_y}\)

and \(b' = \bar{X} - a'\bar{Y}\)

Remark: The two lines \((D_{y,x})\) and \((D'_{x,y})\) are concurrent at G.

**Coefficient of Correlation:**

This is the measure of relationship between the two variables \(X\) and \(Y\).

The coefficient is denoted by \(r\) and is equal to: \(r = \frac{\text{Cov}(X;Y)}{\sigma_X \sigma_Y}\)

Interpretation:

If \(0 < r < 0.5\) then there is a weak positive correlation between \(X\) and \(Y\).

If \(0.5 < r < 0.75\) then there is an average positive correlation between \(X\) and \(Y\).

If \(0.75 < r < 1\) then there is a strong positive correlation between \(X\) and \(Y\).

If \(-0.5 < r < 0\) then there is a weak negative correlation between \(X\) and \(Y\).

If \(-0.75 < r < -0.5\) then there is an average negative correlation between \(X\) and \(Y\).

If \(-1 < r < -0.75\) then there is a strong negative correlation between \(X\) and \(Y\).

**Important percentages:**

Percentage of increase of a variable = \(\frac{F-I}{I} \times 100\)

Percentage of decrease of a variable = \(\frac{I-F}{I} \times 100\)

\(F\): The final value of the variable.

\(I\): Initial value of the variable.

Percentage of error in the estimation of a variable: \(\frac{|R-E|}{R} \times 100\)

\(R\): The real value of the variable.

\(E\): The estimated value of the variable.
**Exercise 5:**
The evolution of the turnover of a company for the period 2001-2010, is given in the table below:

<table>
<thead>
<tr>
<th>Year</th>
<th>2001</th>
<th>2003</th>
<th>2006</th>
<th>2008</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank of the year $x_i$</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>Turnover expressed in billions of dollars $y_i$</td>
<td>64.8</td>
<td>68.7</td>
<td>72.7</td>
<td>77.1</td>
<td>82.1</td>
</tr>
</tbody>
</table>

1) Calculate the means $\bar{x}$ and $\bar{y}$
2) Represent in an orthogonal system, the scatter plot of the points associated to this series as well as the mean point G.
3) Determine by using the least squares method, the equation of the regression line $(D_{y/x})$, of $y$ in $x$, then draw the line in the same system.
4) Estimate the turnover in the year 2012.
5) Starting from which year would the turnover of this company exceed 100 billion of LL?
6) Calculate the percentage of the increase of turnover from the year 2006 to the year 2010.

**Exercise 6:**
The evolution of the number of monitors produced by a grand enterprise starting from the year 1995 is given by the following table:

<table>
<thead>
<tr>
<th>Year</th>
<th>1995</th>
<th>1997</th>
<th>1999</th>
<th>2000</th>
<th>2002</th>
<th>2004</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank of the year $x_i$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>Number of monitors (in hundreds) $y_i$</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>28</td>
<td>30</td>
<td>32</td>
</tr>
</tbody>
</table>

1) Calculate the percentage increase of the number of monitors from the year 1995 to the year 2004
2) Represent in an orthogonal system, the scatter plot of the points $(x_i, y_i)$ associated to this series.
3) Calculate the coordinates of the center of gravity G and plot this point in the preceding system.
4) Calculate the correlation coefficient and give a meaning of the value thus obtained.
5) Write the equation of the regression line $(D_{y/x})$ of $y$ in $x$ and draw it in the same system.
6) Suppose that the model evolution remains valid until 2015.
   a) Estimate the number of monitors produced in 2009.
   b) In reality the number of monitors produced in 2009 is 4500 monitors. Calculate, in percentage, the error in the preceding estimation.
   c) Estimate the year in which the enterprise has produced 2785 monitors.
   d) In which year would the number of produced monitors exceed 5000 for the first time?
   e) The average cost of production of a monitor is 800$. Each monitor is sold for 1000$. Estimate in which year the benefit of the enterprise was 710600$
**Exercise 7:**

**Part A:**

The table below represents the cost $Y$, expressed in thousands of L.L, of $X$ units expressed in hundreds, of a certain product.

<table>
<thead>
<tr>
<th>$X_i$</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>15</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_i$</td>
<td>8</td>
<td>16</td>
<td>19</td>
<td>21</td>
<td>23</td>
<td>24</td>
</tr>
</tbody>
</table>

1) Calculate the mean $\bar{x}$ and $\bar{y}$ of the two variables $X$ and $Y$
2) Represent in an orthogonal system, the scatter plot of the points $(X_i, Y_i)$ associated to this series and the center of gravity $G$ in the same system.
3) Write the equation of the regression line $(D_{y/x})$ of $y$ in $x$ and draw it in the same system.
4) Estimate the cost of production of 4000 units.
5) Calculate the minimum number of units to produce for the cost to exceed 5000 L.L

**Part B:**

The table below represents the revenue $Z$, expressed in thousands of L.L, of the same quantity of the preceding product.

<table>
<thead>
<tr>
<th>$X_i$</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>15</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_i$</td>
<td>7</td>
<td>15</td>
<td>17</td>
<td>23</td>
<td>25</td>
<td>28</td>
</tr>
</tbody>
</table>

Knowing that the regression line $(D_{z/x})$ of $z$ in $x$, is given by: $z = 1.04x + 7.16$
1) Determine the coordinates of the intersection point $L$ of the lines $(D_{y/x})$ and $(D_{z/x})$
2) Give an economical interpretation to the coordinates of $L$.

**Exercise 8:**

The fuel consumption $y$, of a car that travels 100 km, is expressed as a function of its speed $x$ as shown in the following table:

<table>
<thead>
<tr>
<th>$x$ (in km/h)</th>
<th>80</th>
<th>90</th>
<th>120</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ (in liters)</td>
<td>5</td>
<td>5.5</td>
<td>8.4</td>
<td>12</td>
</tr>
</tbody>
</table>

1) Represent the scatter plot of the points $(x_i, y_i)$ in a rectangular system.
2) Calculate the averages $\bar{x}$ and $\bar{y}$. Place the center of gravity $G$ in the preceding system.
3) Determine an equation of the regression line $D_{y/x}$ of $y$ in terms of $x$ and draw this line in the same system.
4) Estimate the fuel consumption of the car that travels 100 km with a speed of 85 km/h.
5) After which speed would the fuel consumption of the car exceed 10 liters for covering 100 km?

**Exercise 9:**

The table below shows the percentage of damaged harvest in a certain village, in the even years 1982, 1984 . . . till 1994.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank of the year $x_i$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>Percentage $y_i$</td>
<td>3.5</td>
<td>3.8</td>
<td>4.6</td>
<td>6.5</td>
<td>6.9</td>
<td>7.8</td>
<td>9</td>
</tr>
</tbody>
</table>

1) Calculate the means $\bar{X}$ and $\bar{Y}$ of the variables $x$ and $y$.
2) Represent graphically the scatter plot of the points $(x_i, y_i)$ as well as the center of gravity $G(\bar{X}, \bar{Y})$ in a rectangular system.
3) Calculate the correlation coefficient $r$ and give an interpretation of the value thus obtained.
4) Determine an equation of $D_{y/x}$, the line of regression of $y$ in terms of $x$, and draw this line in the preceding system.
5) Suppose that the above pattern remains valid till the year 2010. Estimate the percentage of the damaged harvest in the year 2002.

6) In fact, the percentage of the damaged harvest was in reality 13 in the year 2002. What is the percentage of error in the preceding estimation?

**Exercise 10:**

The production of shirts, in a certain factory during the last six years, is distributed as shown in the following table:

<table>
<thead>
<tr>
<th>Rank of the year $x_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production $y_i$ (in thousands)</td>
<td>34.6</td>
<td>35.8</td>
<td>38.8</td>
<td>40.5</td>
<td>41.5</td>
<td>46.1</td>
</tr>
</tbody>
</table>

The line $D_{y/x}$ of regression, of $y$ in terms of $x$, has the equation $y = 2.18x + b$.

1) Determine the coordinates of the point $G$, the center of gravity (mean point) of the scatter plot associated to the given distribution $(x_i ; y_i)$, and deduce the value of $b$.
2) Estimate the production of this factory for the year of rank 10.
Exercise 11:

The results of a survey conducted by a company on the evolution of the price $x_i$ of an article and the number of customers who purchased this article $y_i$, are given by the following table:

<table>
<thead>
<tr>
<th>Unit price of the article: $x_i$ in thousands of LL</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>22</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of purchasers: $y_i$ in hundreds</td>
<td>16</td>
<td>13</td>
<td>11</td>
<td>9</td>
<td>6</td>
</tr>
</tbody>
</table>

1) Construct, in an orthogonal system, the scatter plot of the points $(x_i, y_i)$.
2) Calculate the coordinates of the center of gravity $G$ and plot this point in the preceding system.
3) Determine the coefficient of correlation and interpret the value thus found.
4) Write an equation of the regression line $(D_{y/x})$, of $y$ in terms of $x$, and plot this line in the preceding system.
5) Suppose that the evolution of the price follows the given pattern and that the price of this article reaches 25,000 LL.
   a) Show that an estimation of the number of purchasers, at this price, is 581.
   b) In this case, suppose that each of these customers bought the article. The production cost of this article is 8000 LL. Estimate, thus, the total profit.
FUNCTIONS OF ECONOMICS AND SOCIAL SCIENCES:

**Straight-line depreciation:**
1. **Depreciation:** is the amount an asset losses in value from its original value.
2. **Assets:** are the items that are bought for operating a certain business.
3. Useful lifetime of an asset is the number of years that asset is expected to be usable.
4. **Straight-line depreciation:** is a technique for calculating the value of an asset after it has been in service for a certain period of time under the assumption that its depreciation is the same from year to year.
5. Depreciable value = C - S
6. Annual depreciation = \( \frac{C - S}{n} \)
7. Total depreciation: \( D(t) = \frac{C - S}{n} t \) for \( 0 \leq t \leq n \)
8. Depreciated value : \( V(t) = C - D(t) \) for \( 0 \leq t \leq n \)

**C:** is the original value of the asset.

**n:** lifetime of the asset is \( n \) years.

**S:** value of the asset at the end of its useful lifetime (Salvage or Scrap value)

**Levels of production:**
The production level for a product is the number of units \( x \) of the product that a manufacturing company produces during a certain time period.

**Costs:**
1. Fixed cost: are the costs that must be paid even if nothing is produced (\( x = 0 \)).
2. Variable cost: are the costs of labor and material. These costs depend on the number of units produced.
3. \( C(x) = \) Total costs = Fixed costs + Variable costs.
4. **Average cost per unit** = \( \tilde{C}(x) = \frac{C(x)}{x} \) for all \( x > 0 \)
5. **Marginal cost** = \( M_c(x) = C'(x) \)

**Revenue:**
1. Revenue = (Price per Unit) \times (Quantity Demanded)
2. Average revenue per unit = \( \bar{R}(x) = \frac{R(x)}{x} \) for all \( x > 0 \)
3. Marginal revenue = \( M_R(x) = R'(x) \)

**Profit:**
1. If all units are sold during the same period, then \( P(x) = R(x) - C(x) \). Thus , we have:
   a. If \( P(x) > 0 \), then the company is operating at a profit.
   b. If \( P(x) < 0 \), then the company is operating at a loss.
   c. If \( P(x) = 0 \), then the company is at break-even level that is the company is neither making money nor losing money.
2. **Average profit per unit** = \( \bar{P}(x) = \frac{P(x)}{x} \) for all \( x > 0 \)
3. **Marginal profit** \( M_p(x) = P'(x) \)
4. The profit is a maximum whenever marginal revenue is equal to the marginal cost.
Price, Supply, and demand:
1. The demand function D gives the quantity D(p) of a product that will be sold at price p during the unit of time.
2. The supply function S gives the quantity S(p) of a product that suppliers are willing to produce or supply at price p during the unit of time.
3. The point of intersection of the demand and supply curves is the point at which we have market equilibrium.

Elasticity of demand:
E(p) at price p is defined by
\[ E(p) = -p\frac{D'(p)}{D(p)} \]

Elastic and inelastic demand:
1. Demand is elastic at \( p_0 \) if \( E(p_0) > 1 \)
2. Demand is inelastic at \( p_0 \) if \( E(p_0) < 1 \)
3. Demand has unit elasticity if \( E(p_0) = 1 \)

Revenue and elasticity of demand:
1. If demand is elastic at p, then the revenue \( R(p) \) decreases as the price p increases.
2. If demand is inelastic at p, the revenue \( R(p) \) increases as the price p increases.
3. If demand has unit elasticity, then the revenue \( R(p) \) is maximum at p.

Exercise 12:
Suppose that a company has a group of 12 identical sport cars. If each of these cars is depreciating at a rate of $2000 per year, what is the rate of depreciation of the whole group?

Exercise 13:
Suppose that new office equipment is purchased for $10000, has a scrap value of $2000 after 4 years.

a) Find the linear function that models the total depreciation \( D(t) \).

b) Find the linear function that models the depreciated value \( V(t) \).

c) What would be the value of the equipment after 1st year, 2nd year, 3rd year, and 4th year?

d) Graph \( V \) of \( 0 \leq t \leq 4 \).

Exercise 14:
Suppose that new living room furniture is purchased for $1000, has a useful life time of five years, and has no scrap value. After how many years will it have a value of $600.

Exercise 15:
A manufacturer of scientific calculators has fixed cost of $300 per week and variable costs of $5 per calculator.
Find the total cost function of producing x calculators per week.
Exercise 16:
A manufacturer of pens finds that the total cost of manufacturing \( x \) printers per week is defined by

\[ C(x) = 800 + 60x + \frac{x^2}{10} \]

a) Sketch the graph of this function.

b) Find the fixed costs of the manufacturer.

c) If each printer sells for $600. Find this revenue from selling \( x \) printers per week.

\[ R(x) = 600x \]

d) Determine the production level that yields the maximum profit per week.

Exercise 17:
Let \( D \) be the demand function defined by \( D(p) = 10 - \frac{p}{2} \)

a) Find the elasticity of demand \( E(p) \).

b) Find \( E(4) \). Is \( D \) elastic or inelastic at \( p = 4 \)? Give an economical interpretation of your answer.

Exercise 18:
The total cost of producing a certain product is a function defined by:

\[ C(x) = 100x^2 + 1300x + 1000 \]

where \( x \) is the number (in thousands) of units produced, and \( C(x) \) is in thousands of dollars. The selling price of each unit is $2000.

a) Determine in terms of \( x \), the revenue function defined by \( R(x) \) in thousands of dollars.

b) What must the production be so that profit is zero.

c) i) What must the production be so that the profit is maximum?

\[ P(x) = R(x) - C(x) \]

ii) Calculate this profit.

Exercise 19:
An enterprise produces shirts. The total cost of production expresses in hundreds of $, is given by

\[ C(x) = x^2 - 3x + 4 \]

where \( x \) is the number of produced shirts in tens.

1) Calculate the fixed costs.

2) Calculate the cost of production of 100 shirts.

3) Calculate the number of produced shirts when the cost of production is 5000 $.

4) Find the average cost function \( \overline{C(x)} \)

5) Determine the number of shirts produced for the average cost to be minimum.

   Calculate this minimum.

6) The selling price of a shirt is 20 $ and suppose that the entire production is sold.

   a) Determine the revenue function \( R(x) \)

   b) Deduce that the profit function is given by : \( P(x) = -x^2 + 5x - 4 \)

   c) Determine the dead points.

   d) For what level of production does the enterprise realize a profit ? Justify.

   e) Determine the number of sold shirts for enterprise to realize a maximum profit. Calculate this maximum.
Exercise 20:

Part A:
Let $f$ be a function defined on $]-\infty, 0[ \cup ]0, +\infty[$ by $(x) = \frac{x}{2} + \frac{8}{x}$. Designate by (C) its representative curve in an orthonormal system $(O, i, j)$.
1) Prove that $f$ is odd. Interpret graphically the result.
2) Determine $\lim_{x \to \infty} f(x)$ and $\lim_{x \to 0} f(x)$. Deduce the equation of an asymptote to (C).
3) Prove that the line (d) of equation $y = \frac{x}{2}$ is an asymptote to (C).
4) Show that $f'(x) = \frac{x^2 - 16}{2x^2}$ then construct the table of variations of $f$.
5) Determine the equation of the tangent (T) to (C) at point of abscissa 2.
6) Draw the curve (C), the tangent (T) and the asymptotes.

Part B:
A factory produces batteries. The total cost, in thousands of L.L., is given by $C(x) = \frac{x^2}{2} + 8$ where $x$ is the number in hundreds of produced batteries ($0 \leq x \leq 5$).
1) Calculate the fixed cost.
2) Determine the average cost function $\bar{C}(x)$
3) Determine the number of produced batteries in order to reach minimum average cost.
4) Suppose that the average revenue function is given by $R(x) = -\frac{4x}{5} + 13$. Determine the profit function $P(x)$.
5) Study the variations of $P$ on $[0,5]$ and construct its table of variations.
6) Prove that the equation $P(x) = 0$ admits a unique solution $\alpha$ such that $0.65 < \alpha < 0.66$
7) Determine the minimum number of batteries to produce so that the factory realizes a gain.

Exercise 21:

Part A:
Let $f$ be a function defined on $[0, +\infty[$ by $f(x) = \frac{10x}{(x+1)^2}$ and let (C) its representative curve in an orthonormal system $(O, i, j)$.
1) Determine $\lim_{x \to +\infty} f(x)$. Deduce the existence of an asymptote to (C).
2) Show that $f'(x) = \frac{10(1-x)}{(1+x)^3}$. Construct the table of variations of the function $f$.
3) Draw the curve (C).

Part B: An enterprise produces radio transistors. The unit price $p$ is expressed in $\$, with $0 \leq x \leq 20$. Designate by $D(p) = \frac{10}{(p+1)^2}$ the demand function and $g(p) = p - \frac{89}{10}$ the supply function, expressed in thousands of transistors.
1) Determine the revenue function $R(p)$. Deduce the unit price for which the enterprise realize maximum revenue and calculate the quantity that was sold under this price.
2) Express the elasticity of the demand $E(p)$ in terms of $p$.
Study the elasticity of the demand for the price of 4$. Give an economical interpretation.
3) Verify that $10p^3 - 69p^2 - 168p - 189 = (p - 9)(10p^2 + 21p + 21)$
4) Find the equilibrium price of the market.
**Composite functions:**
Let \( f \) and \( g \) be two function defined on \( I \) and \( J \) respectively such that \( f(I) \subseteq J \). The composition \( f \circ g \) and \( g \) is a function denoted by \( g \circ f(x) = g(f(x)) \) (we read it \( g \) of \( f(x) \)).

**Inverse functions:**
1) Given the function \( f \) continuous and strictly monotone over \( I \) then \( f \) admits an inverse function \( f^{-1} \) over \( I \).
2) The curve \( (C) \) of \( f \) and the curve \( (C') \) of \( f^{-1} \) are symmetric with respect to the line \( y = x \).
3) **Asymptotes:**

<table>
<thead>
<tr>
<th>To the Curve ( (C) ) of ( f )</th>
<th>To the curve ( (C') ) of ( f^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = a ) is vertical asymptote</td>
<td>( y = a ) is horizontal asymptote</td>
</tr>
<tr>
<td>( y = b ) is horizontal asymptote</td>
<td>( x = b ) is vertical asymptote</td>
</tr>
<tr>
<td>( y = ax + b ) is an oblique asymptote</td>
<td>( x = ay + b ) is an oblique asymptote</td>
</tr>
</tbody>
</table>

4) If the point \( A(x_A; y_A) \in (C) \) then \( A'(y_A; x_A) \in (C') \).
5) Domain of \( f^{-1} = \)Range of \( f \).
6) \( f \circ f^{-1}(y) = y \) and \( f^{-1} \circ f(x) = x \).

**Derivative of inverse function:**
\( (f^{-1})'(y) = \frac{1}{f'(x)} \) where \( y = f(x) \).

**Primitives of a functions:**
Let \( f \) be a function defined and continuous on an interval \( I \). We say that \( F \) is a primitive of \( f \) on \( I \), if and only if \( F \) is derivable and \( F'(x) = f(x) \) for every \( x \in I \).

**Antiderivative:**
If \( F \) is a primitive of \( f \) on an interval \( I \), we note:
\( \int f(x)dx = F(x) + C \).

**Properties:**
1) \( \int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \)
2) \( \int k \cdot f(x)dx = k \int f(x)dx \) where \( k \) is a real constant.

**Anti-derivatives of some functions:**
1) \( \int 0dx = C \)
2) \( \int kdx = kx + C \)
3) \( \int xdx = \frac{1}{2}x^2 + C \)
4) \( \int x^ndx = \frac{x^{n+1}}{n+1} + C \)
5) \( \int \frac{1}{x}dx = -\frac{1}{x} + C \)
6) \( \int u^n(x)u'(x)dx = \frac{u^{n+1}(x)}{n+1} + C \)
7) \( \int u^1(x)u'(x)dx = \frac{u^2(x)}{2} + C \)
8) \( \int \frac{u'(x)}{u(x)}dx = \ln(u(x)) + C \) \( u(x) > 0 \)
9) \( \int \frac{1}{x}dx = \ln x + C \)
10) \( \int e^xdx = e^x + C \)
11) \( \int u'(x)e^{u(x)}dx = e^{u(x)} + C \)
12) \( \int \frac{u'(x)}{u^2(x)}dx = \frac{-1}{u(x)} + C \)
Definite integrals:
If $F$ is a primitive of $f$ on an interval $I$ and if $[a; b]$ is a subset of $I$. We note $\int_a^b f(x)dx = F(b) - F(a)$ and we call it the defined integral of $f$ on $[a; b]$.

Properties:
1) $\int_a^a f(x)dx = 0$ and $\int_a^0 0dx = 0$
2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4) $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5) $\int_a^b kf(x)dx = k \int_a^b f(x)dx$
6) If $f \geq 0$ then $\int_a^b f(x)dx \geq 0$
7) $f \leq g$ then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
8) If $f$ is even then $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$
9) If $f$ is odd then $\int_{-a}^a f(x)dx = 0$

Integration by parts:
$\int_a^b u'(x)v(x)dx = [u(x)v(x)]_a^b - \int_a^b u(x)v'(x)dx$

Exercise 22:
Show that $F$ is an antiderivative of $f$ over $R$, in each of the following cases:

1) $F(x) = x^2 - 3$ and $f(x) = 2x$
2) $F(x) = \frac{x^3}{5} + \frac{4}{3}x^2 - 8x + 3$ and $f(x) = \frac{3x^2}{5} + \frac{8x}{3} - 8$
3) $F(x) = x^2 - 3x$ and $f(x) = 2x - 3$

Exercise 23:

Determine the following anti-derivatives (primitives)

1) $\int xdx$
2) $\int 2dx$
3) $\int 3x^3dx$
4) $\int \frac{2}{x^2}dx$
5) $\int (x^2 - 2x + 7)dx$
6) $\int (\frac{1}{x^3} + 2x)dx$
7) $\int \frac{2x^2 + 3x}{x}dx$
8) $\int (x + 1)^2dx$
9) $\int (2x - 3)(x^2 - 3x + 2)dx$
10) $\int x(x^2 - 1)dx$
11) $\int 3(x - 2)^2dx$
12) $\int \left( x - \frac{3}{x^3} \right)dx$

Exercise 24:

Determine an antiderivative $F$ of the continuous function $f$ on $I$ verifying the given condition:

1) $f(x) = 4x$ \hspace{1cm} $I = R$ and $F(1) = 3$
2) $f(x) = -x^2 + 2x - 5$ \hspace{1cm} $I = R$ and $F(1) = 0$
3) \( f(x) = \frac{1}{x^2} \) \\
\( l = ]-\infty;0[ \) and \( F(-1) = \frac{2}{5} \)

**Exercise 25:**

Calculate the following definite integrals:

1) \( \int_{1}^{9} dx \) 
2) \( \int_{-2}^{4} 2xdx \) 
3) \( \int_{0}^{3} (x^2 + 3x)dx \) 
4) \( \int_{-2}^{2} -3xdx \) 
5) \( \int_{0}^{1} (9x^5 + 3x^3 - 2x)dx \) 
6) \( \int_{1}^{3} \frac{2}{x^3}dx \) 
7) \( \int_{0}^{4} (x^2 + x - 1)dx \) 
8) \( \int_{1}^{3} \frac{x^2 + 2x}{x^2}dx \) 
9) \( \int_{-2}^{1} (2x + 1)^3dx \)

**Exercise 26:**

a) Represent graphically the function \( f \) defined on \( \mathbb{R} \) by \( f(x) = 1 - x^2 \)

b) Calculate the area of the region bounded by the curve \( (C_f) \) of \( f \), the x-axis and the two lines of equations \( x = -1 \) and \( x = 1 \).

**Exercise 27:**

a) Represent graphically the line defined by equation \( f(x) = -2x + 8 \)

b) Calculate the area bounded by this line, the x-axis and the two lines of equations \( x = 1 \) and \( x = 6 \)

**Exercise 28:**

Let \( f \) be a function defined over \( \mathbb{R} \) by \( f(x) = x^2 + 1 \) and its representative curve is given below:

Calculate the area of the shaded region?
Logarithmic function:

Properties of logarithmic functions:

Given that $a > 0$ and $b > 0$

1) $\ln(ab) = \ln(a) + \ln(b)$
2) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
3) $\ln\left(\frac{1}{a}\right) = -\ln(a)$
4) $\ln(e^x) = x$
5) $\ln(1) = 0$
6) $\ln x > 0$ for $x > 1$
7) $\ln x < 0$ for $0 < x < 1$

Derivative of logarithmic functions

$$\left[\ln w(x)\right]' = \frac{w'(x)}{w(x)}$$

Limits of logarithmic functions:

$$\lim_{x \to 0^+} \ln x = -\infty$$
$$\lim_{x \to +\infty} \ln x = +\infty$$
$$\lim_{x \to 0^+} x^\alpha \ln x = 0^-$$
$$\lim_{x \to +\infty} \frac{\ln x}{x^\alpha} = 0$$

where $\alpha$ is strictly positive real number.

Properties about $\ln$:

$\ln a = \ln b$ than $a = b$
$\ln a < \ln b$ than $a < b$
$\ln a > \ln b$ than $a > b$

Solving equations:

$\ln x = k$ ($x > 0$) then $\ln x = \ln e^k$ so $x = e^k$

Exercise 29:

Simplify the following expressions:

1) $2 \ln(16) - 3 \ln(10) + \ln(125) + 2\ln(2.5)$
2) $2 \ln(\sqrt{e}) - \ln\left(\frac{1}{e^2}\right) + \ln(e^{-2}) + 4 \ln(\sqrt{e^3}) + (\ln e)^2$
3) $\ln\left(\frac{3\sqrt{7}+2}{7}\right) + \ln\left(\frac{3\sqrt{7}-2}{2}\right)$
Exercise 30:
Determine the domain of definition of each of the following functions:
1) \( f(x) = \ln(x - 1) \)  
2) \( f(x) = \ln(x + 1) + \ln(4 - x) \)  
3) \( f(x) = \frac{\ln(x-1)}{x^2} \)  
4) \( f(x) = \ln\left(\frac{3-x}{x+1}\right) \)  
5) \( f(x) = \ln[(x+2)(3-x)] \)  
6) \( f(x) = \ln(4 - x^2) \)  
7) \( f(x) = \frac{1+\ln x}{1-\ln x} \)  
8) \( f(x) = \ln(1 - \ln x) \)

Exercise 31:
Solve the following equations and inequalities:
1) \( \ln(2x - 1) = 0 \)  
2) \( \ln(x + 3) = \ln(3 + 5x) \)  
3) \( \ln(3 - 2x) - \ln x = 2 + \ln 3 \)  
4) \( \ln(x^2 + 2x - 3) = 2\ln 6 - \ln 3 \)  
5) \( \ln^2 x - 5\ln\left(\frac{1}{x}\right) + 3 = \ln x \)  
6) \( \ln\left(\frac{x-1}{x+1}\right) = -1 \)  
7) \( \ln(3x - 7) < 0 \)  
8) \( \ln(5x - 4) \geq 2\ln x \)  
9) \( \ln(x - 3) + \ln(3x - 2) \geq \ln 10 \)  
10) \( (1 - \ln x)(2 + \ln x) \geq 0 \)  
11) \( \ln(1 - \ln x) > 0 \)  
12) \( 2\ln^2 x - \ln x > 0 \)

Exercise 32:
Solve in \( \mathbb{R} \), the following systems:
1) \( \begin{cases} \ln x + \ln y = 3 \\ 2\ln x + 5\ln y = 9 \end{cases} \)  
2) \( \begin{cases} \ln x - \ln y = \ln 4 \\ x + y = 5 \end{cases} \)  
3) \( \begin{cases} \ln x \ln y = 10 \\ \ln(xy) = 7 \end{cases} \)  
4) \( \begin{cases} \ln x + \ln y = 1 \\ (\ln x)(\ln y) = -2 \end{cases} \)

Exercise 33:
Calculate the derivative of each of the following functions:
1) \( f(x) = 1 + \frac{1}{x} + \ln x \)  
2) \( f(x) = x + 2\frac{\ln x}{x} \)  
3) \( f(x) = \frac{\ln x + 2}{2\ln x + 1} \)  
4) \( f(x) = \ln^2 x - 3\ln x + 4 \)  
5) \( f(x) = \ln(1 - \ln x) \)  
6) \( f(x) = \ln(x^2 - 4x + 5) \)  
7) \( f(x) = \frac{x\ln x}{x - 1} \)  
8) \( f(x) = x^2 + 3x\ln x \)  
9) \( f(x) = (1 + 2\ln x)^4 \)  
10) \( f(x) = \ln\left(\frac{x-2}{x+2}\right) \)

Exercise 34:
Calculate the following limits:
1) \( \lim_{x\to\infty} \frac{3 + 2\ln x}{1 + \ln x} \)  
2) \( \lim_{x\to\infty} (x^2 + 2 - \ln(x+1)) \)  
3) \( \lim_{x\to\infty} (x^2 + \ln(x+1) - \ln x) \)  
4) \( \lim_{x\to\infty} (2x(\ln^2 x - 3\ln x + 4)) \)  
5) \( \lim_{x\to\infty} \frac{x - \ln x}{x + \ln x} \)  
6) \( \lim_{x\to\infty} (2x + \ln\left(\frac{x-2}{x+2}\right)) \)
Exercise 35:
Study the variations and draw the curve of each of the following functions:

1) \( f(x) = \frac{\ln x}{x} \)
2) \( f(x) = x^2 + \ln x \)
3) \( f(x) = x^2 - 2\ln x \)
4) \( f(x) = x\ln x \)
5) \( f(x) = 1 + \frac{\ln x}{x^2} \)
6) \( f(x) = \ln^2 x \)

Exercise 36:
Let \( f \) be the function on \( ]0; +\infty[ \) by \( f(x) = 1 - \frac{1}{x} + \ln x \) and let (C) be its representative curve in an orthonormal system \((O, \overrightarrow{i}, \overrightarrow{j})\)

1) Calculate limits of \( f \) at the boundaries of the domain.
2) Calculate \( f'(x) \) and set up the table of variations of \( f \).
3) Determine the equation of the tangent (T) to (C) at the point of abscissa 1.
4) Draw (T) and (C).
5) Let \( F \) be the function defined on \( ]0; +\infty[ \) by \( F(x) = (x - 1)\ln x \)
   a) Show that \( F \) is a primitive of \( f \).
   b) Calculate the area of the domain limited by the curve (C), the axis of abscissas and the lines of equations \( x = 1 \) and \( x = e \)

Exercise 37:
Let \( f \) be the function on \( ]0; e[ \) by \( f(x) = \ln(1 - \ln x) \) and let (C) be its representative curve in an orthonormal system \((O, \overrightarrow{i}, \overrightarrow{j})\)

1) Calculate the limits of \( f \) at the boundaries of its domain of definition. Deduce the equations of the asymptotes.
2) Show that \( f \) is strictly decreasing and construct its table of variations.
3) Draw the curve (C) and its asymptotes.
4) Prove that \( f \) admits an inverse function \( g \).
Determine the domain of definition of \( g \) and indicate its asymptotes then draw its curve \((C')\).

Exercise 38:
The function \( f \) and \( g \) defined by \( f(x) = \ln x \) and \( g(x) = \frac{1}{x} \)
1) Determine the domain of definition of \( f \); \( g ; f \circ g \) and \( g \circ f \).
2) Calculate \( f'(x) ; g'(x) ; (f \circ g)'(x) \) and \((g \circ f)'(x)\) in terms of \( x \)
3) Express \((f \circ g)(x)\) and \((g \circ f)(x)\) in terms of \( x \)
4) Find again \( (f \circ g)'(x) \) and \( (g \circ f)'(x) \) in terms of \( x \).

**Exercise 39:**

The curve (C) here is the representative curve, in an Orthonormal system, of the function \( f \) defined on \( ]0; +\infty[ \) by \( f(x) = \ln(ax + b) \) where \( a \) and \( b \) are Constants to be determined.

The curve (C) admits at \( O \) a tangent (T) of equation \( y = x \). Show that \( f(x) = \ln(x + 1) \).

**Exercise 40:**

**Part A:** The curve \( (C_g) \) here is that of the function \( g \) defined on \( ]0; +\infty[ \) by \( g(x) = x + a + b \ln x \) where \( a \) and \( b \) are two real constants.

1) Determine graphically \( g(1) \) and \( g'(2) \).
2) Calculate \( g'(x) \) in terms of \( b \) and \( x \).
3) Show that \( a = 1 \) and \( b = -2 \).
4) Indicate why we have \( g(x) > 0 \).

**Part B:** Let \( f \) be a function defined on \( ]0; +\infty[ \) by \( f(x) = x^{\frac{3}{2} + 3 - 2 \ln x} \) and let (C) be its representative curve in an orthonormal system \((O, \vec{i}, \vec{j})\).

1) Calculate \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to +\infty} f(x) \). Deduce an asymptote to the curve (C).
2) Show that \( f'(x) = g(x) \). Set up the table of variations of \( f \).
3) Show that the curve (C) admits an inflection point to be determined.
4) Draw the curve (C).

**Exercise 41:**

**Part A:**

Let \( f \) be a function defined on \( ]1, +\infty[ \) by \( f(x) = \frac{1 + \ln x}{\ln x} \) and let (C) be its representative curve in an orthonormal system \((O, \vec{i}, \vec{j})\).

1) Calculate Limits of \( f \) at the boundaries of the domain.
2) Show that \( f'(x) = \frac{-1}{x(\ln x)^3} \) and set up the table of variations of \( f \).
3) Determine the equation of the tangent (T) to (C) at the point of abscissa \( e \).
4) Draw (C), (T) and the asymptotes.
5) The line of equation \( y = x + 1 \) cuts (C) at a point of abscissa \( \alpha \). Verify that \( 1.7 < \alpha < 1.8 \).

6) Prove that \( f \) admits on \( ]1; +\infty[ \) an inverse function \( g \).

Determine the domain of definition of \( g \) and indicate its asymptotes then draw its curve.

7) Express \( g(x) \) in terms of \( x \).

**Part B:** (In what follows, take \( \alpha = 1.75 \))

An enterprise fabricates a certain product. The functions of demand and supply are respectively

\[ f(p) = \frac{1+\ln p}{ln p} \quad \text{and} \quad g(p) = p + 1, \]

where \( f(p) \) and \( g(p) \) are expressed in hundreds of units and \( p \) is the unit price in thousands of L.L such that \( 1.1 \leq p \leq 6 \).

1) Calculate the demand if the price of each unit is 4000 L.L

2) Determine the price if the demand is 500 units.

3) Determine the price of equilibrium.

4) Determine the elasticity of the demand \( E(p) \).

5) Calculate the needed price for the demand to have a unit elasticity.

6) Calculate \( E(4) \). Is the demand elastic for \( p=4 \)? Give an economical interpretation.

**Exercise 42:**

**Part A:** Let \( f \) be a function defined on \( ]-1; +\infty[ \) by

\[ f(x) = ax + 4 - 3\ln(x + b) \]

and let (C) be its representative curve in an orthonormal system.

1) Knowing that (C) passes through the points \( A(0;4) \) and \( B(e - 1; e) \), show that \( a = b = 1 \).

2) Calculate \( \lim_{x \to -1^-} f(x) \) and \( \lim_{x \to +\infty} f(x) \). Deduce an asymptote to (C).

3) Calculate \( f'(x) \) and set up the table of variations of \( f \).

4) Draw the curve (C).

**Part B:** For a grand company of architecture, the cost of construction of \( x \) hundreds of houses is given by \( C(x) = x + 4 - 3\ln(x + 1) \), where \( x \in [0; 20] \) and \( C(x) \) is in millions of dollars.

1) Calculate \( C(10) - C(0) \). Interpret the result.

2) What is the number of houses that minimize the cost? Calculate the minimum cost.

3) The selling price of house is 200000 $.

a) Prove that the revenue function is \( R(x) = 2x \).

b) Deduce the profit function has the equation: \( P(x) = x - 4 + 3\ln(x + 1) \).

b) Does the company realize a gain when it sells 100 houses? Justify your answer.

4) Following figure represents the curve of the function of profit \( P \) defined on \( [0; 20] \).

a) Prove that the equation \( P(x) = 0 \) admits a unique solution \( \alpha \). Verify that \( 1.3 < \alpha < 1.4 \).

b) Deduce the minimum number of houses to be produced.
so that the company realizes a gain.

**Part C:**
An enterprise produces televisions where \( x \) is the unit price in Hundreds of dollars with \( f(x) = x + 4 - 3 \ln(x + 1) \) and \( g(x) = -0.5x + 5 \) are the functions of supply and demand respectively in ten thousands of television; \( x \in [0; 20] \)

1) Calculate the demand if the price of a television is $600.
2) Prove graphically, that the equation \( f(x) = g(x) \) admits a unique solution \( \beta \). Verify that \( 3.8 < \beta < 3.9 \)
3) Suppose that \( \beta = 3.85 \). Give an economical interpretation to the values of \( \beta \) and \( f(\beta) \)
4) An employee of this enterprise said if the price of a television is $500, then the enterprise cannot assure the needs of the market. Is he right? Justify.
5) Prove that the elasticity of the demand is \( E(x) = \frac{-x}{x-10} \)
6) Study the elasticity of the demand for the price of $500. Is the demand elastic for this price? Give an economical interpretation to the result.

**Exercise 43:**

**Part A:** Let \( f \) be the function defined on \( ]e, +\infty[ \) by \( f(x) = \frac{1 - \ln x}{1 + \ln x} \) and let \( (C) \) be its representative curve in an orthonormal system \((O, i, j)\).

1) Calculate \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to \frac{1}{e}} f(x) \)
2) Verify that \( f'(x) < 0 \) and set up the table of variations of \( f \).
3) Write the equation of the tangent \((d)\) to \((C)\) at the point of abscissa \(1\).
4) Draw \((d)\) and \((C)\).
5) Prove that \( f \) admits an inverse function \( f^{-1} \) over \( ]e, +\infty[ \). Determine the domain of definition of \( f^{-1} \) and indicates its asymptotes and set up its table of variations.
6) Show that \( A(-0.5, e^3) \) is a point of \((C_{f^{-1}})\) then Express \( f^{-1}(x) \) in terms of \( x \).

**Part B:** An enterprise produces electric batteries where the unit price \( p \) is expressed in thousands of LL; \((0.4 \leq p \leq 2.7)\). The demand \( f(p) \) of this product, expressed in thousands of units, is given by: \( f(p) = \frac{1-ln p}{1+ln p} \)

1) Calculate the number of electric batteries demanded for a unit price of 1500LL.
2) Find the elasticity \( e(p) \) of the demand with respect to the price.
Exercise 44:

Part A:

Let \( f \) be the function defined on \( ]0, \infty[ \) by:

\[
f(x) = ax + b \ln x.
\]

and by \( (C) \) its representative Curve in an orthonormal system \((O, \overrightarrow{i}, \overrightarrow{j})\).

As indicates the given figure \((T)\) is tangent to \((C)\) at the point of abscissa 1.

1) Determine graphically \( \lim_{x \to 0^+} f(x) \), \( f(1) \), \( f'(1) \)

and \( \lim_{x \to +\infty} f(x) \)

2) Determine \( f'(x) \) in terms of \( a \) and \( b \)

3) Show that \( a = 1 \) and \( b = 2 \)

4) Prove that the line \((d)\) of equation \( y = x \) is an asymptote to \((C)\).

Study the relative position of \((C)\) and \((d)\).

5) Set up the table of variations of \( f \).

6) Prove that the equation \( f(x) = 0 \) admits unique solution \( \alpha \) such that \( 0.75 < \alpha < 0.76 \)

7) Calculate the area of the domain limited by the curve \((C)\), the line \((d)\) and the lines of equation \( x = 1 \) and \( x = e \)

Part B:

An enterprise produces watches. The function \( C(x) = x + 2 \ln x \) represents the average cost of production in millions of L.L., where \( x \) is the numbers of watches produced in thousands \((1 \leq x \leq 10)\)

1) Calculate \( C(3) \). Give an economical interpretation to the found result.

2) Express \( C(x) \) in terms of \( x \)

3) Each watch is sold for 50000L.L and all production is sold. Prove that the revenue function is \( R(x) = 5x \)

4) Deduce that \( P(x) = 2x - x^2 - 2\ln x \)

5) Study the variations of \( P(x) \) and construct its table of variations on \([1; 10]\)

6) Deduce the number of watches that the enterprise has to sell in order to achieve maximum profit and calculate this maximum.
Numerical sequence:

**Sense of Variation of sequence:**

If \( U_{n+1} - U_n \geq 0 \) then \( U_n \) is increasing sequence

If \( U_{n+1} - U_n \leq 0 \) then \( U_n \) is decreasing sequence

**Rules of arithmetic and Geometric sequence:**

**Arithmetic sequence:**
1) To show a sequence \( U_n \) is arithmetic you have to prove that \( U_{n+1} - U_n = d \) where \( d \) is called the common difference.
2) To show a sequence \( U_n \) is not arithmetic you have to show that \( U_2 - U_1 \neq U_1 - U_0 \) or \( U_3 - U_2 \neq U_2 - U_1 \) or ..............
3) To write \( U_n \) in terms of \( n \):
   \[ U_n = U_0 + (n - p)d \]
   in particular \( U_n = U_0 + nd \)
4) Sum of arithmetic sequence:
   \[ S = U_0 + U_1 + U_2 + \ldots + U_n = \frac{\text{number of terms}}{2} \times (\text{initial term} + \text{final term}) \]

**Geometric sequence:**
1) To show a sequence \( U_n \) is geometric sequence you have to prove that \( \frac{U_{n+1}}{U_n} = r \) where \( r \) is called common ratio.
2) To show a sequence \( U_n \) is not geometric sequence you have to prove that:
   \[ \frac{U_2}{U_1} \neq \frac{U_1}{U_0} \text{ or } \frac{U_3}{U_2} \neq \frac{U_2}{U_1} \text{ or } \ldots \]
3) To write \( U_n \) in terms of \( n \):
   \[ U_n = U_0(r)^{n-p} \]
   in particular \( U_n = U_0 r^n \)
4) Sum of geometric sequence:
   \[ S = U_0 + U_1 + U_2 + \ldots + U_n = \text{initial term} \times \frac{1 - r^{n+1}}{1 - r} \]

**Exercise 45:**

Study the sense of variation of the sequence \( (U_n) \) in each of the following cases :\( (n \geq 0) \)

1) \( U_n = 2n + 3 \)
2) \( U_n = -n^2 - 2n + 3 \)
3) \( U_n = \frac{n+3}{n+2} \)
4) \( U_n = e^n \)
5) \( U_n = 3(0.5)^n \)
6) \( U_{n+1} = 3 + U_n \)

**Exercise 46:**
Determine the real number $x$ if $a, b, c$ are three consecutive terms of an arithmetic sequence:

$a = x$ ; $b = 2x + 1$ and $c = 4x - 1$

**Exercise 47:**

Calculate the tenth term of an arithmetic sequence $U_n$ of first term $U_1 = 3$ and such that $U_4 = 27$

**Exercise 48:**

Calculate the common difference of an arithmetic sequence $(U_n)$ of first term $U_0 = 5$ and $U_5 = 25$

**Exercise 49:**

$U_n$ is an arithmetic sequence. Calculate $U_0$ if $U_1 = -7$ and $U_8 = -28$

**Exercise 50:**

Determine the real number $x$ if $a, b, c$ are three consecutive terms of a geometric sequence:

$a = x + 1$ ; $b = 3x - 1$ and $c = 5x + 1$

**Exercise 51:**

Calculate the fourth term of a geometric sequence $U_n$ of first term $U_1 = 3$ and $U_3 = 27$

**Exercise 52:**

Calculate the ratio of a geometric sequence $(U_n)$ of first term $U_0 = 4$ and such that $U_2 = 1$

**Exercise 53:**

$(U_n)$ is a geometric sequence. Calculate $U_0$ if $U_1 = 6$ and $U_3 = 54$

**Exercise 54:**

Calculate the sum $S$ in each of the following cases:

1) $S = 1 + 2 + 3 + \cdots + n$  
2) $S = 5 + 7 + 9 + \cdots + 61$  
3) $S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$  
4) $S = \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{729}$

**Exercise 55:**

Let $(U_n)$ be a sequence defined by $U_0 = 2$ and $U_{n+1} = \frac{U_n}{4} - 3$. ($n \geq 0$)

1) Calculate $U_1$ and $U_2$.
2) Show that $(U_n)$ is neither arithmetic nor geometric.
3) We define the sequence $(V_n)$ by $V_n = 2U_n + 8$ for every ($n \geq 0$)
   a) Show that $(V_n)$ is geometric sequence whose common ratio and first term $V_0$ are to be determined.
   b) Study the sense of variations of $(V_n)$
   c) Express $(V_n)$ in terms of $n$. Deduce $U_8$
d) Express $U_{n+1} - U_n$ in terms of $n$. Deduce the sense of variations of $(U_n)$

4) Let $S_n = U_0 + U_1 + \cdots + U_n$ and $S'_n = V_0 + V_1 + \cdots + V_n$
   a) Calculate $S'_n$ in terms of $n$.
   b) Deduce the expression of $S_n$ in terms of $n$, as well as $\lim_{n \to +\infty} S_n$

**Exercise 56:**

Let $(U_n)$ be a sequence defined by $U_0 = 7$ and $U_{n+1} = \frac{U_n + 12}{4}$.

Suppose that $V_n = U_n - 4$ ($n \geq 0$)

1) Show that $(V_n)$ is geometric sequence whose common ratio and first term are to be determined

2) Express $V_n$ then $U_n$ in terms of $n$.

3) Study the sense of variations of the sequence $(U_n)$ and $(V_n)$

4) Calculate $U_0 + U_1 + \cdots + U_n$ in terms of $n$.

5) Let $(W_n)$ be a sequence defined by $W_n = \ln(V_n)$
   a) Show that $(W_n)$ is an arithmetic sequence whose common difference and first term are to be determined.
   b) Express $W_n$ in terms of $n$.
   c) Calculate $W_0 + W_1 + \cdots + W_n$ in terms of $n$.

**Exercise 57:**

Let $(U_n)$ be a sequence defined by:

\[
\begin{cases}
U_0 = 2 \\
U_{n+1} = \frac{5U_n - 1}{U_n + 3} ; \quad n \geq 0
\end{cases}
\]

1) Calculate $U_1$ and $U_2$

2) Let $(V_n)$ be a sequence defined by: $V_n = \frac{1}{U_n - 1}$

Show that $(V_n)$ is an arithmetic sequence whose common difference and first term $V_0$ are to be determined.

3) Express $V_n$ in terms of $n$. Deduce that $U_n = \frac{n + 8}{n + 4}$

4) Calculate $U_5$ and give the value of $S = 2 + \frac{9}{5} + \cdots + \frac{13}{9}$

**Exercise 58:**

A cyclist runs every day for a one hour. Every day, he improves his performance by running a distance that is 1% more than the distance he ran the previous day.

1) The first day, the cyclist ran 41 km. What is the distance that he will run in the second day?

2) Designate by $(U_n)$ the distance in Km, ran during the nth day and let $U_1 = 41$

a) Express $U_n$ in terms of $n$.

b) Calculate the distance ran by the cyclist on the 21st day.

c) Starting from which day would the distance ran by the cyclist exceed 60 km.

d) Calculate the distance ran by the cyclist during the first three weeks.
Exercise 59:
An employee receives the amount of 2000$ each month. On the first day, he spends 20% of this amount on the second day he spends 20% of the amount remaining with him from the previous day, and so on for every new day. Designate by $U_n$ the amount, in LL, left with this employee at the end of the nth day. ($n \geq 1$)

1) Verify that $U_1 = 1600$
2) Prove that $(U_n)$ is a geometric sequence whose ratio is to be determined.
3) Calculate $U_n$ in terms of $n$.
4) At the end of which day, the amount left with this employee would become for the first time less than 500$

Exercise 60:
Part A:
Let $(U_n)$ be a sequence defined by $U_{n+1} = \frac{24}{25} U_n and U_0 > 0$

1) Calculate $U_1 and U_2 in terms of $U_0$
2) Prove that $(U_n)$ is geometric sequence whose ratio is to be determined.
3) What is the sense of variations of $(U_n)$
4) Express $U_n$ in terms of $U_0$ and $n$.

Part B:
On the first of January 1997, a certain population is constituted of 3000 persons. Suppose that this population decreases by 4% each year.

1) Determine the size of this population on the first of January 1999.
2) In which year this population becomes less than 2000 persons for the first time?

Exercise 61:
The manager of a sports club declares that, every year, 75% of the members of the club renew their membership and 800 new members join the club. In 2005, the club had 1600 members. ($U_0 = 1600$).

Denote by $U_n$ the number of members in the club in the year (2005+n).

1) Verify that $U_1 = 2000 and calculate $U_2$.
2) Show that $U_{n+1} = 0.75U_n + 800$, for every natural integer $n$.
3) Consider the sequence $(V_n)$ defined by $V_n = 3200 - U_n$
   a) Show that $(V_n)$ is a geometric sequence. Specify its common ratio and its first term.
   b) Express $V_n$ in terms of $n$.
4) Assuming that the development of the number of members continues according to the given pattern, is it possible for the number of members to double?
5) Show that \( U_{n+1} - U_n = 400(0.75)^n \). Deduce the sense of variation of \((U_n)\).
6) Calculate \( U_{10} \) and interpret its value.
7) In which year the number of members becomes greater than 2500 members for the first time?

**Exercise 62:**

**Part A:**

On January 1, 2009, Sami deposited 2000000 L.L., in a saving account, that earns compound interest at the annual interest rate of 6% compounded monthly. Moreover, he added to this account, at the beginning of every month starting from February 1, 2009, an amount of 500 000 L.L. Designate by \( U_n \) the amount in Sami’s account at the beginning of the \( n \)th month; Thus \( U_0 = 2000000 \)

1) Prove that \( U_1 = 2510000 \)
2) Justify that \( U_{n+1} = 1.005U_n + 500000 \)
3) Consider the sequence \((V_n)\) defined by \( V_n = U_n + \alpha \) where \( \alpha \) is a real constant.
   a) Calculate \( \alpha \) for which the sequence \((V_n)\) is geometric sequence with common ratio 1.005.
   b) Calculate \( V_n \) then \( U_n \) in terms of \( n \).
4) Determine the date at which Sami had, in his account, for the first time, an amount greater than or equal to 20000000 L.L.

**Part B:**

Sami wants to buy a car costs 20000000 L.L. He was offered the choice to pay this amount in 5 monthly payments that are the five consecutive terms of an arithmetic sequence with common difference 100000.

Calculate the amount of each of these 5 monthly payments.

**Exercise 63:**

An owner proposes, in 1/1/1990, an apartment with the initial annual rent of 30000$ and he offered two types of increase:

1) In the first case, the rate of increase of this rent is 3% each year. Designate by \( U_n \) the annual amount of the rent for the year \((2010+n)\) and by \( U_0 = 30000 \)
   a) Calculate the amount of the rent for the year 2011.
   b) Show that \((U_n)\) is a geometric sequence whose common ratio is to be determined. Express \( U_n \) in terms of \( n \).
   c) What would be the amount of the rent in 2015?
   d) In which year, would the rent be more than the double of the initial rent?
   e) What would be the sum perceived by the owner along the first twenty years?
2) In the second case, the annual rent increases by 1000$ every year. Designate by \( V_n \) the annual rent for the year \((2010+n)\) and by \( V_0 = 30000 \$\)
   a) Show that \((V_n)\) is arithmetic sequence. Express \( V_n \) in terms of \( n \).
   b) What would be the sum perceived by the owner along the first twenty years?
3) Which of the two contracts is more advantageous for the tenant over the period of 20 years?
Exercise 64:
Zahi deposits a capital \( C_0 = 10\,000\,000 \) LL in an investment company.
At the end of every year, this company transfers into Zahi’s account an interest of 5% together
with a supplementary amount of 200 000 LL.
Designate by \( C_n \) the balance in his account at the end of the \( n \)th year.

1) Verify that \( C_1 = 10\,700\,000 \) LL.

2) Prove that \( C_{n+1} = (1.05)C_n + 200\,000 \).

3) Consider the sequence \( (S_n) \) defined by \( S_n = C_n + 4\,000\,000 \); \( n \geq 0 \).
   a) Prove that \( (S_n) \) is a geometric sequence of ratio 1.05 and calculate \( S_0 \).
   b) Write \( S_n \) in terms of \( n \), and deduce \( C_n \) in terms of \( n \).
   c) Find the number of years needed for the balance in Zahi’s account, in this company, to exceed 17 000 000 LL for the first time?

Exercise 65:
A statistical study of the population of a certain village revealed the following information:
- The population of this village was 6000 at the beginning of the year 2000.
- The annual increase in the population of this village is 2%.
- 200 persons leave this village permanently every year (moving to the town, immigrating to other countries).

Designate by \( U_n \) the number of inhabitants in this village in the year \((2000 + n)\).

1) Let \( U_0 = 6000 \), verify that \( U_1 = 5920 \).

2) Show that \( U_{n+1} = 1.02U_n - 200 \).

3) Consider the sequence \( (V_n) \) that is defined by \( V_n = U_n - 10\,000 \); \( n \geq 0 \).
   a) Prove that \( (V_n) \) is a geometric sequence of common ratio 1.02.
   b) Calculate \( V_n \) in terms of \( n \) and deduce \( U_n \) in terms of \( n \).
   c) During which year would the number of inhabitants in this village become less than 3000 for the first time?

Exercise 66:
A merchant borrows a loan of \( 20\,000\,000 \) LL from a certain bank.
The annual rate of interest charged is 6%, compounded monthly.
To pay back this loan, he decides to pay \( 500\,000 \) LL to the bank at the end of every month.
Designate by \( U_n \) the amount of the debt at the end of the \( n \)th month.

1) Verify that \( U_1 = 19\,600\,000 \).

2) Establish that \( U_{n+1} = 1.005U_n - 500\,000 \).

3) Consider the sequence \( (V_n) \) that is defined by \( V_n = U_n - 100\,000\,000 \).
   a) Prove that \( (V_n) \) is a geometric sequence of ratio 1.005 and determine \( V_1 \).
b) Express $V_n$ in terms of $n$, and deduce $U_n$ in terms of $n$.

c) Prove that this debt is paid back at the end of 45 months.

d) Determine the value of the last amount that is to be paid by the merchant at the end of the 45th month.

Exercise 67:
Fadi deposits a capital of 100 million LL in a bank, at 10% annual interest rate, compounded yearly. At the end of every year, Fadi withdraws 5 million LL from his account.
Let $U_0 = 100$ and designate by $U_n$ the amount, in millions LL, that is in Fadi’s account at the end of the nth year after withdrawing the 5 million LL.
1) a) Verify that $U_1 = 105$ and calculate $U_2$.
   b) Show that the sequence $(U_n)$ is not geometric.
   c) Justify the relation $U_{n+1} = 1.1U_n - 5$.
2) Let $V_n = U_n - 50$, for every natural integer $n$.
   a) Show that the sequence $(V_n)$ is a geometric sequence of common ratio 1.1.
   b) Calculate $V_n$ in terms of $n$, and find the value of $U_8$.

Exercise 68:
A person rented an apartment at the beginning of the year 2000.
The annual rent in the year 2000 was 4 000 000 LL to be increased by 10% every year.
Let $U_0 = 4 000 000$ and designate by $U_n$ the annual rent in LL in the year $(2000 + n)$.
1) Calculate $U_1$ and $U_2$.
2) a) Prove that $(U_n)$ is a geometric sequence, and determine its common ratio.
   b) Calculate $U_n$ in terms of $n$, and deduce $U_{n+1}$ in terms of $n$.
3) Let $S_n = U_0 + U_1 + \ldots + U_n$.
   a) Show that $1.1 \times S_n = U_1 + U_2 + \ldots + U_{n+1}$.
   b) Deduce that $1.1 \times S_n = S_{n+1} - U_0$, and that $S_n = 40 000 000 \left( (1.1)^{n+1} - 1 \right)$.
4) This person rented the apartment for 6 consecutive years starting from the beginning of the year 2000 till the end of 2005.
   Calculate the total sum of money paid by this person for renting this apartment during this period.

Exercise 69:
A bank proposes to its customers, who are younger than 25 years, the following offer:
Depositing an amount of 2 000 000 LL in an account at an annual interest rate of 9% compounded monthly to which the bank adds directly and each month an amount of 9000 LL.
Imad decides to take advantage of this offer.
Denote by $S_n$ the amount in Imad’s account after $n$ months. Thus, $S_0 = 2 000 000$.
1) Prove that $S_{n+1} = 1.0075 S_n + 9000$.
2) $(V_n)$ is the sequence defined by $V_n = S_n + 1 200 000$ for all natural numbers $n$.
   a) Show that $(V_n)$ is a geometric sequence whose common ratio and first term $V_0$ are to be determined.
   b) Express $V_n$ in terms of $n$. Deduce $S_n$ in terms of $n$.
3) After how many months will the amount in Imad’s account exceed 4 000 000 LL for the first time?
Exponential functions:

Properties of exponential functions

1) \( e^x > 0 \forall x \in \mathbb{R} \)
2) \( e^a e^b = e^{a+b} \)
3) \( \frac{e^a}{e^b} = e^{a-b} \)
4) \( e^1 = e \)
5) \( e^0 = 1 \)

Derivative exponential functions:

\((e^{u(x)})' = u'(x) e^{u(x)} \quad (e^x)' = e^x\)

Primitive of exponential function:

\( \int u'(x) e^{u(x)} \, dx = e^{u(x)} + c \) in particular \( \int e^x \, dx = e^x + c \)

Limits of exponential functions:

\( \lim_{{x \to -\infty}} e^x = 0 \) and \( \lim_{{x \to +\infty}} e^x = +\infty \)

Solving equations:

a) \( \ln x = k \quad x > 0 \)
   If \( k = 0 \) then \( x = 1 \)
   If not then \( e^{k \ln x} = e^k \) so \( x = e^k \)

b) \( e^x = k \)
   If \( k < 0 \) or \( k = 0 \) then no solution because \( e^x > 0 \)
   If \( k > 0 \) then \( \ln e^x = \ln k \) so \( x = \ln k \)

Exercise 70:
Simplify the following expressions:

1) \( e^{\ln 2} \)
2) \( e^{-\ln 2} \)
3) \( e^{0.5 \ln 16} \)
4) \( e^{2 \ln 3 - 3 \ln 2} \)
5) \( e^{\ln x} - \ln e^x \)
6) \( e^{\frac{2 + \ln 32}{e^{3 \ln 8}}} \)
7) \( \ln \left( \frac{e^{1+x}}{e} \right) - \ln \left( \frac{1}{e^{2x}} \right) \)
8) \( \ln (e^x + 1) - \ln (e^{-x} + 1) \)
9) \( (e^x + e^{-x})^2 + (e^x - e^{-x})^2 \)

Exercise 71:
Find the domain of definition of each of the following functions:

1) \( f(x) = x + e^x \)
2) \( f(x) = \frac{e^x}{x} \)
3) \( \frac{x-1}{e^x} \)
4) \( f(x) = \frac{e^x + x}{e^x - 1} \)
5) \( f(x) = \ln (e^x + 1) \)
6) \( \ln \left( e^x - 1 \right) \)
7) \( f(x) = e^x \ln x \)
8) \( f(x) = \ln \left( \frac{1 + e^x}{1 - e^x} \right) \)
Exercise 72:
Solve the following equations and inequalities:
1) \( e^x = 0 \)
2) \( e^{x+1} = 2 \)
3) \( e^{3x+4} = e^{x+2} \)
4) \( e^{1-x} = 2e^x \)
5) \( (e^x - 1)(e^x + 4) = 0 \)
6) \( (x - 1)e^{\ln(x+2)} = 2x + 4 \)
7) \( e^{2(x+1)} + 3e^{x+2} = 4e^2 \)
8) \( \frac{8}{e^{x+1}} = e^x - 1 \)
9) \( e^x - 1 - 6e^{-x} = 0 \)
10) \( e^x > x \)
11) \( \frac{1-e^{-x}}{e^x} \geq 0 \)
12) \( 4e^{2x} - 5e^x + 1 > 0 \)
13) \( \ln(e^x + 1) \leq 2 \)
14) \( \frac{e^{-x} - 5}{1-e^{-x}} \geq -1 \)

Exercise 73:
Solve in \( \mathbb{R} \), the following systems:
1) \( \begin{cases} e^x + e^y = 3 \\ 2e^x + 5e^y = 9 \end{cases} \)
2) \( \begin{cases} e^x - e^y = 1 \\ e^x e^y = 2 \end{cases} \)
3) \( \begin{cases} e^x = e^{lny} \\ e^x + y = \frac{e^4}{e^2} \end{cases} \)
4) \( \begin{cases} e^x \times e^y = e^5 \\ lnx + lny = ln6 \end{cases} \)

Exercise 74:
Calculate the derivative of each of the following functions:
1) \( f(x) = x + 1 + 2e^x \)
2) \( f(x) = -2x + e^{-x} + 4 \)
3) \( f(x) = e^x - e^{-x} \)
4) \( f(x) = x + 1 - xe^{-x} \)
5) \( f(x) = \frac{1-e^x}{1+e^x} \)
6) \( f(x) = x - 2 - \frac{4}{e^x+1} \)
7) \( f(x) = x + 2 - \frac{3e^x}{e^x+3} \)
8) \( f(x) = 6(x-2)e^{-0.5x} - 1 \)
9) \( f(x) = \ln(e^x + 1) \)
10) \( f(x) = (1 + 2e^x)^5 \)
11) \( e^{-(x^2 - 2x)}e^{-x} \)
12) \( f(x) = \frac{xe^x}{e^x+1} \)
13) \( f(x) = \frac{e^{x}e^{-x}}{e^x - e^{-x}} \)
14) \( e^x lnx \)

Exercise 75:
Calculate the following limits:
1) \( \lim_{x \to \infty} (1 + e^x + e^{-x}) \)
2) \( \lim_{x \to \infty} (e^{2x} + e^x + 4) \)
3) \( \lim_{x \to \infty} (e^{2x} + e^x + 4) \)
4) \( \lim_{x \to \infty} (e^{-2x} + e^{-x} + 4) \)
5) \( \lim_{x \to \infty} (e^{-2x} + e^{-x} + 4) \)
6) \( \lim_{x \to \infty} \frac{e^x - 2}{e^x + 2} \)
7) \( \lim_{x \to \infty} \frac{e^x}{x} \)
8) \( \lim_{x \to \infty} \frac{(x+1)e^{-x}}{x^3} \)
9) \( \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \)
10) \[ \lim_{x \to 0^+} \frac{1-e^x}{x} \]

11) \[ \lim_{x \to \infty} (x^2 + x + 1 - e^x) \]

12) \[ \lim_{x \to \infty} \left( \frac{\ln(e^x + 1)}{e^x} \right) \]

13) \[ \lim_{x \to \infty} (x + 1 - xe^x) \]

14) \[ \lim_{x \to \infty} 6(x-2)e^{-0.5x} - 1 \]

15) \[ \lim_{x \to \infty} (x + 1 - x^2 e^{-x}) \]

**Exercise 76:**

Study the variations and draw the curve of each of the following functions:

1) \( f(x) = 1 + e^x \)  
2) \( f(x) = (x + 2)e^x \)  
3) \( f(x) = 1 - e^{-x} \)

4) \( f(x) = x + e^{-x} \)  
5) \( f(x) = xe^{-x} \)  
6) \( f(x) = e^x - e^{-x} \)

**Exercise 77:**

Let \( (x) = (x + 2)e^{-x} \) and \( F(x) = (ax + b)e^{-x} \); \( x \in \mathbb{R} \) where \( a \) and \( b \) are two real constants.

Calculate \( a \) and \( b \) if \( F \) is primitive (anti derivative) of the function \( f \).

**Exercise 78:**

**Part A:** Let \( f \) be a function defined over \( \mathbb{R} \) by \( f(x) = a + (b - x)e^x \) and let \( (C) \) be its curve in an orthonormal system \((0, \overrightarrow{i}, \overrightarrow{j})\).

Determine \( a \) and \( b \) if the curve \( (C) \) passes through the point \( A(0;3) \) and admits at this point a tangent parallel to the line \((D)\) of equation \( y = x \).

**Part B:**

Suppose that \( f(x) = 1 + (2 - x)e^x \)

1) Calculate \( \lim_{x \to +\infty} f(x) \) and write \( f(2.5) \) in a decimal form.

2) Calculate \( \lim_{x \to -\infty} f(x) \). Deduce an asymptote \((d)\) to \((C)\). Study the relative position of \((C)\) and \((d)\).

3) Calculate \( f'(x) \) and set up the table of variations of \( f \).

4) Prove that the equation \( f(x) = 0 \) admit a unique solution \( \alpha \). Verify that \( 2.1 < \alpha < 2.2 \)

5) Draw \((d)\) and \((C)\).

6) Let \( F \) be a function defined on \( \mathbb{R} \) by \( F(x) = x + (p + qx)e^x \)

   a) Calculate the area of the domain bounded by \((C)\), the axis of abscissas and the two lines of equations \( x = 0 \) and \( x = 1 \).
Exercise 79:

Part A:
Let $g$ be a function defined by $g(x) = e^x - x - 1$
1) Calculate $\lim_{x \to -\infty} g(x)$ and $\lim_{x \to \infty} g(x)$
2) Calculate $g'(x)$ and set up the table of variations of $g$.
3) Deduce the sign of $g(x)$ on $\mathbb{R}$.

Part B:
Let $f$ be a function defined on $\mathbb{R}$ by $f(x) = x + \frac{x+2}{e^x}$ and let $(C)$ be its representative curve in an orthonormal system $(O, i, j)$.
1) Calculate $\lim_{x \to -\infty} f(x)$
2) Calculate $\lim_{x \to +\infty} f(x)$. Prove that the line $(D)$ of equation $y = x - 2$ is an asymptote to $(C)$ to $(C)$ and study the relative position of $(C)$ and $(D)$.
3) Verify that $f'(x) = \frac{g(x)}{e^x}$. Set up the table of variations of $f$.
4) Calculate the coordinates of the point $A$ of $(C)$ where the tangent $(T)$ is parallel to $(D)$.
5) Prove that the equation $f(x) = 0$ admits a unique solution. Verify that $-1.69 < \alpha < -1.68$
6) Draw $(C)$, $(D)$ and $(T)$.
7) a) Show that $x \to (x - 3)e^{-x}$ is the primitive of $x \to (x + 2)e^{-x}$
   b) Calculate the area of the domain limited by $(C)$, the axis of abscissas and the line of equations $x = 0$ and $x = 1$

Exercise 80:

Part A:
Let $f$ be the function that is defined on $[-1;+\infty[$ by $f(x) = x - 2 - 2xe^{x}$, and let $(C)$ be its representative curve in an orthonormal system $(O; i, j)$.
1) a) Calculate $\lim_{x \to +\infty} f(x)$, and prove that the line $(d)$ of equation $y = x - 2$ is an asymptote of $(C)$
   b) Study, according to the values of $x$, the relative positions of $(C)$ and $(d)$.
   c) Calculate $f(0)$ and $f(-1)$.
2) Given below the table of signs of $f'(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-1$</th>
<th>$0.3$</th>
<th>$+\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Set up the table of variations of $f$.
3) a) Draw $(d)$ and $(C)$.
   b) Show, graphically, that the equation $f(x) = 0$ has a unique positive solution $\alpha$.
   Verify that $2.4 < \alpha < 2.5$.

Part B: In all what follows, suppose that $\alpha = 2.45$.
A factory produces a certain chemical liquid.
The function $M_C$, defined on $[0;10]$ by $M_C(x) = 1 + 2(1 - x)e^{-x}$, expresses the daily marginal cost of this production.

$x$ is expressed in thousands of liters, and $M_C(x)$ in millions LL.

The fixed cost of this production amounts to 2 million LL.

1) Prove that the total cost function $C$ is expressed by $C(x) = x + 2 + 2xe^{-x}$.

2) The whole production is completely sold at the price of 2000 LL per liter.
   a) Prove that the profit function is expressed by $P(x) = x - 2 - 2xe^{-x}$
   b) Determine the quantity that should be produced daily by this factory in order that the profit is zero.
   c) Does the factory achieve a profit if the daily production of this liquid is 2000 liters? Justify your answer.

**Exercise 81:**

**Part A:**

Let $f$ be the function that is defined, on $[0; +\infty[$, by $f(x) = 3(x + 1)e^{-x}$ and let $(C)$ be its representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$.

1) Calculate $\lim_{x \to +\infty} f(x)$ and determine an asymptote of (C).
2) Show that $f'(x) = -3xe^{-x}$ and set up the table of variations of $f$.
3) Draw the curve (C).
4) Let $F$ be the function that is defined on $[0; +\infty[$ by $F(x) = 3(-x - 2)e^{-x}$.
   a) Show that $F$ is an antiderivative (a primitive) of $f$.
   b) Calculate the area of the region bounded by the curve (C), the axis of abscissas and the lines of equations $x = 0$ and $x = 1$.

**Part B:**

A factory produces a certain chemical liquid. The demand is modeled by:

$f(p) = 3(p + 1)e^{-p}$; where $p$ is the unit price expressed in thousands LL and $f(p)$ is expressed in thousands of liters, for $0.5 \leq p \leq 4$.

1) Calculate the demand corresponding to a unit price of 1000 LL.

2) The supply is modeled by $g(p) = \frac{e^p}{3}$

The adjoining curve (T) is the representative curve of the function $h$ defined $h(p) = f(p) - g(p)$, on $[0.5; 4]$.

3) a) Verify that the equation $h(p) = 0$ has a unique root $\alpha$ and prove that: $1.57 < \alpha < 1.58$.
   b) Suppose that $\alpha = 1.575$.

Give an economical interpretation of this value of $\alpha$.

4) a) Calculate $E(p)$, the elasticity of the demand with respect to the price $p$.
   b) Determine the set of values of $p$ for which the demand is elastic,
and find the corresponding prices.

**Exercise 82:**

Let \( f \) be the function that is defined, on \([ 0 ; + \infty [\), by: \( f(x) = x + \frac{1}{2} + e^{1-x} \)

and designate by \((C)\) its representative curve in an orthonormal system \((O; \mathbf{i}, \mathbf{j})\).

**Part A:**

1. **a)** Calculate \( \lim_{{x \to +\infty}} f(x) \).

   \[ b) \] Prove that the line \((d)\) of equation \( y = x + \frac{1}{2} \) is an asymptote of \((C)\).

2. Calculate \( f'(x) \) and set up the table of variations of \( f \).

3. Draw \((d)\) and \((C)\).

4. Calculate the area of the region that is bounded by the curve \((C)\), its asymptote \((d)\) and the two lines of equations \( x = 0 \) and \( x = 1 \).

**Part B:**

A factory manufactures batteries and the total cost of production, in millions LL, is expressed by \( C(x) = x + \frac{1}{2} + e^{1-x} \) where \( x \) is the number, in hundreds, of batteries produced \((0 \leq x \leq 5)\).

1. Calculate the fixed costs.

2. Calculate the total cost of manufacturing 500 batteries.

3. Each battery is sold for 20 000 LL, but only 90% of the production is sold.
   a) Show that the revenue function is expressed by \( R(x) = 1.8x \).

   \[ b) \] Represent graphically the function \( R \), in the system \((O; \mathbf{i}, \mathbf{j})\).

   c) Justify graphically that the equation \( R(x) = C(x) \) has a unique solution \( \alpha \) and verify that \( 1.43 < \alpha < 1.44 \).

   d) What does \( \alpha \) represent to the factory?

   e) Indicate the minimal number of batteries that should be manufactured in order that the factory achieves a profit.
Exercise 83:

Part A: The curve (C) to the right is the graphical representation, in an orthonormal system, of the function $f$ defined over $[0 ; + \infty [$ by $f(x) = a e^x + b$, where $a$ and $b$ are real numbers.

The curve (C) has at $O$ a tangent (T) of equation $y = \frac{x}{8}$.

Show that $f(x) = e^x - \frac{1}{8}$.

Part B:

Let $g$ be the function defined over $[0 ; + \infty [$ by $g(x) = \frac{120}{e^x + 15}$, and let (G) be its representative curve.

1) Determine $\lim_{x \to +\infty} g(x)$ and deduce an asymptote to (G).

2) Show that $g'(x) < 0$, for every $x \geq 0$.

3) Set up the table of variations of $g$.

4) Copy (C) and draw (G) in the same orthonormal system.

5) Suppose that $f$ and $g$ are respectively the supply and the demand functions of a certain object, in terms of the unit price $x$. ($x$ is expressed in millions LL, $f(x)$ and $g(x)$ in hundreds of objects).

   a) Estimate the number of objects demanded when the unit price is $5 \ 000 \ 000$ LL.

   b) Show that the equation $f(x) = g(x)$ has a unique solution $\alpha$ and verify that $\alpha = \text{ln}(25)$.

   c) Calculate $g(\alpha)$ and give an economical interpretation for the values of $\alpha$ and $g(\alpha)$. 
Combinations:

The number of combinations of \( r \) elements among \( n \) elements is given by:

\[
C_n^r = \frac{n!}{r!(n-r)!}
\]

Permutations without repetition:

The number of permutation without repetition of \( r \) elements among \( n \) elements is given by:

\[
P_n^r = \frac{n!}{(n-r)!}
\]

Permutations with repetition:

The number of permutations with repetition of \( r \) elements among \( n \) elements is given by \( n^r \)

Exercise 84:
Simplify the following:

1) \( \frac{(n+2)!}{n!} \)
2) \( \frac{C_n^2 	imes C_n^4}{C_n^2} \)
3) \( \frac{C_n^3}{C_n^2} \)
4) \( \frac{P_n^4}{P_n^3} \)

Exercise 85:
Solve the following equations:

1) \( C_n^2 = 15 \)
2) \( C_n^7 = C_n^5 \)

Exercise 86:
A bus holds 20 seats. 6 persons took that bus. In how many ways can these person sit?

Exercise 87:
In an aquarium, there are three red fish, two black and one silver. We draw, simultaneously and at random, three fish to put them in a vase.

1) How many possibilities are there?
2) How many possibilities are there if the three chosen fish are of three different colors?

Exercise 88:
Six girls and five boys want to form a volleyball team. The number of players of this game is 6.

1) What is the number of possible teams?
2) What is the number of possible teams formed of four girls and two boys.

Exercise 89:
An assembly of 20 persons want to elect a committee of four members: a president, a vice-president, a treasurer and a secretary. Successively

1) What is the number of possible committees?
2) What is the number of possible committees if the president must be the oldest (supposed unique)?
3) Suppose that these 20 persons are 15 women and 5 men. It was decided that the positions of the president and the treasurer will be occupied by women, while the vise-president and that secretary will be occupied by men. What is, in this case, the number of possible committees?

Exercise 90:
A student has to answer 12 questions among 20 questions in an exam.
1) What is the number of possible choices?
2) What is the number of choices if the first two questions are obligatory?
3) What is the number of choices if he has to answer 7 questions among the first 12 questions and 5 questions among the remaining questions.

Exercise 91:
A bag contains five tokens: one blue token worth 3 points, two red tokens each worth 1 point, two green tokens each worth 1 point. We draw one by one and without replacement two tokens. Calculate the number of possible results to:
1) Obtain two tokens of different colors.
2) Obtain 2 points
3) Obtain 2 points with two tokens of different colors.
4) Obtain at least 2 points.

Exercise 92:
An urn contains 4 red balls, 5 white balls and 3 black balls.

Part A:
We draw simultaneously three balls from the urn.
1) What is the number of possible drawings.
2) What is the number of drawings of:
   a) Three white balls.
   b) Three balls of the same color?
   c) No red balls?
   d) At least one red ball
   e) One black ball only?
   f) Three balls of three different colors.
   g) At least two white balls
   h) At most one red ball?
   i) At most two red balls.

Part B:
We draw successively and without replacement three balls from the urn.
1) What is the possible drawings.
2) What is the number of drawings of:
   a) Three white balls
   b) Three balls with the same color.
   c) No red balls?
   d) At least one red ball?
   e) The first ball is the only black ball
   f) Only one black ball?
   g) One red ball, one black ball and one white in this order.
   h) Three balls of three different colors.

Exercise 93:
Dima and Samer are two members of a group of 17 men and 12 women. We want to choose 6 of these members to form a committee.

1) How many committees can we form.
2) What is the number of committees which:
   a) Contain Dima and Samer?
   b) Do not contain neither Dima nor Samer?
c) Contain Dima or Samer?
d) Contain Dima only?

**Probability:**

1) \( P(\overline{A}) = 1 - P(A) \)
2) \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
3) i) A and B are called mutually exclusive if \( A \cap B = \emptyset \)
   
   ii) A and B are mutually exclusive then \( P(A \cup B) = P(A) + P(B) \)

4) Conditional probability:
   
   i) Probability of A given that B is occurred is denoted by \( P(A/B) \)
   
   where \( P(A/B) = \frac{P(A \cap B)}{P(B)} \)
   
   ii) \( P(B/A) = \frac{P(A \cap B)}{P(A)} \)

5) A and B are independent iff:
   
   i) \( P(A/B) = P(A) \)
   
   ii) \( P(B/A) = P(B) \)
   
   iii) \( P(A \cap B) = P(A)P(B) \)

6) \( P(A) = P(A \cap B) + P(A \cap \overline{B}) \)

**Exercise 94:**

An urn contains six balls: one blue ball, two red balls and three green balls.

1) We draw, simultaneously and at random, 2 balls from the urn.
   
   a) What is the probability of drawing two red balls?
   
   b) What is the probability of drawing two balls having the same color?

2) We draw, one by one without replacement, 2 balls from the run.
   
   a) What is the probability of drawing one blue ball and one red ball in this order?
   
   b) What is the probability of drawing one blue ball and one red ball?

3) We draw, one by one with replacement, 2 balls from the urn.
   
   a) What is the probability of drawing two balls with different colors?
   
   b) What is the probability that the second drawn ball is the only green ball?

**Exercise 95:**

An urn contains 9 balls: \( m \) white, \( n \) red and 2 black.

We draw, successively and without replacement, 3 balls from the urn.

1) a) What is the number of possible drawings?

   b) Calculate \( m \) and \( n \) if the probability of drawing a white ball and two black balls is equal to \( \frac{1}{21} \)

2) Suppose that \( m = 4 \) and \( n = 3 \). What is the probability of drawing three white balls?
3) What is the probability that the third drawn ball is the only white ball among the three drawn balls.
4) What is the probability of drawing a ball of each color?
5) What is the probability of drawing at least a white ball and a red ball in the same drawing?

**Exercise 96:**
Suppose that A and B are two events with \( P(A) = 0.3 \) and \( P(B) = 0.6 \) Find \( P(A \text{ or } B) \) if the two events are:
- a) Incompatible (Mutually exclusive)
- b) Independent

**Exercise 97:**
Suppose that E and F are two events for which \( P(E) = 0.3 \), \( P(F) = 0.7 \) and \( P(E \text{ or } F) = 0.79 \)
- a) Are E and F mutually exclusive?
- b) Are E and F independent?

**Exercise 98:**
A bag contains seven balls:
- A red ball carrying the number 2
- Two yellow balls each carrying the number 3
- Four green balls each carrying the number 4.
Two balls are drawn successively and without replacement, from this bag.
1) Calculate the probability of drawing red ball followed by a green ball.
2) Calculate the probability of drawing two green balls.
3) Calculate the probability of drawing two balls having the same color.
4) Calculate the probability of drawing two balls where the product of the numbers they carry is a multiple of 4.

**Exercise 99:**
Suppose that for all the days of September, the probability that it rains is \( \frac{1}{4} \)
If it rains, the probability that Mister X arrives on time to his work is \( \frac{1}{3} \). If it doesn’t rain, the probability that Mister X arrives on time to his work is \( \frac{5}{6} \).
1) What is the probability, a given day in September, it rains and Mister X arrives on time to his work.
2) Calculate the probability that, a given day in September, Mister X arrives on time to his work.
3) A day in September, Mister X arrives on time to his work. What is the probability that it rained that day?
4) Over a period of four days of September, what is the probability that Mister X arrives at least once on time to his work?

**Exercise 100:**
To compose a jury, we choose at random 8 persons among 40. Sami and Chaza are among the 40 persons. Calculate the probability so that:
1) Sami is a part of the jury.
2) Sami and Chaza is a part of the jury.
3) Neither Sami nor Chaza is a part of the jury.
4) At least one of the two is a part of the jury.
5) One is apart of the jury and the other is not.
6) Sami and Chaza are not in the same jury.

**Exercise 101:**
The probability that a student studies is 0.75. Given that he studies, the probability that she will pass the final examination is 0.8. Given that she doesn’t study, the probability that she will pass the final exam is 0.25. Find the probability that:

a) She will study and pass the final exam.

b) She will not study and pass the final exam.

c) She will not pass the final exam

**Exercise 102:**
The 40 students in the basketball club of a school are distributed as shown in the table below:

<table>
<thead>
<tr>
<th></th>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st secondary year</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>2nd secondary year</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>3rd secondary year</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>

A group of 3 students is chosen simultaneously and randomly from this club.

1) Consider the following events:
   A: « The three chosen students are all in the 3rd secondary year ».
   B: « The three chosen students are all girls, each one from a different year ».
   C: « The three chosen students are all in the same secondary year ».

Verify that the probability $p(A)$ is equal to $\frac{7}{152}$ and calculate $p(B)$ and $p(C)$.

2) The chosen group is formed of three boys, what is the probability that they are all from the same secondary year?

3) In this part we choose randomly and successively three students from this club.
   a) What is the probability that the first is in the 1st year, the second is in the 2nd year and the third is in the 3rd year.
   b) What is the probability that at least one of them is in the 1st year.

**Exercise 103:**
In order to prevent a certain disease, we vaccinated 40% of persons of a population. Then we noticed that 85% of the vaccinated persons were not affected by the disease and that 75% of the persons who were not vaccinated are affected by the disease.

A person is chosen randomly from this population.

Consider the following events:
   D: « the chosen person is affected by the disease».
   V: « the chosen person is vaccinated ».

1) a) Verify that the probability of the event $D \cap V$ is equal to $\frac{6}{100}$.

b) What is the probability that the chosen person is affected by the disease and is not vaccinated?
c) Deduce the probability $P(D)$.
2) The chosen person is not affected by the disease.
   Calculate the probability that he/she is vaccinated.
3) In this part, suppose that this population is formed of 300 persons.
   We choose randomly 3 persons from this population.
   What is the probability that at least one, among the 3 chosen persons, is vaccinated?

Exercise 104:
An urn $U$ contains 2 red balls and 3 black balls.
An urn $V$ contains 3 red balls and 2 black balls.
We draw at random a ball from urn $U$. If the ball is black we place it in the urn $V$; If not we put it out side of the two urns. Then we draw at random a ball from the urn $V$.
Consider the following events:
   $R_1$: The drawn ball from urn $U$ is red.
   $B_1$: The drawn ball from urn $U$ is black.
   $R_2$: The drawn ball from urn $V$ is red.
   $B_2$: The drawn ball from urn $V$ is black.

1) 
   a) Calculate $P(R_1)$ and $P(B_1)$
   b) Calculate $P(R_2 | R_1)$ and $P(R_2 | B_1)$
   c) Deduce that $P(R_2) = \frac{27}{50}$

2) Calculate $P(B_2)$
   Let $X$ be the random variable which is equal to the number of red balls obtained in the process of the two drawings.
   a) Determine the law of probability of $X$.
   b) Calculate the expected value $E(X)$.

Exercise 105:
A library has 100 calculators distributed according to type and year of manufacture as shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Type P</th>
<th>Type G</th>
<th>Type O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufactured in 2007</td>
<td>20</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>Manufactured in 2006</td>
<td>10</td>
<td>12</td>
<td>18</td>
</tr>
</tbody>
</table>

A) A customer chooses at random one of these calculators.
1) Knowing that the chosen calculator was manufactured in 2007, show that the probability that it is of type $G$ is equal to 0.25.
2) What is the probability that the chosen calculator is of the type $O$ and manufactured in 2007?
3) The prices of the calculators are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Type P</th>
<th>Type G</th>
<th>Type O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufactured in 2007</td>
<td>100 000 LL</td>
<td>80 000 LL</td>
<td>60 000 LL</td>
</tr>
<tr>
<td>Manufactured in 2006</td>
<td>50 000 LL</td>
<td>40 000 LL</td>
<td>30 000 LL</td>
</tr>
</tbody>
</table>
B) In this part, the customer chooses randomly and simultaneously two out of these 100 calculators.
1) What is the probability that the two chosen calculators are manufactured in 2007?
2) What is the probability that the price of the two chosen calculators is 180 000 LL?
3) What is the probability that the price of the chosen calculator does not exceed 70 000 LL.

**Exercise 106:**
In a gift shop, pens are displayed on a shelf. These pens are of two different brands E and F. 40% of these pens are of brand E. 25% of the pens of brand E are golden. \( \frac{1}{3} \) of the pens of brand F are golden.

A) A client chooses a pen at random from this shelf.
1) What is the probability of choosing a golden pen of brand E?
2) What is the probability of choosing a golden pen?
3) The chosen pen is golden. What is the probability that it is of brand E?

B) In this part suppose that 20 pens are displayed on the shelf.
A client chooses, simultaneously and at random, three pens from these 20 pens.
1) Show that the probability of choosing exactly two pens of brand E is \( \frac{28}{95} \).
2) The price of a pen of brand E is 250 000 LL and that of a pen of brand F is 150 000 LL. Designate by X the random variable equal to the sum paid by this client for buying the three chosen pens.
   a) Determine the four possible values of X.
   b) Determine the probability distribution of X.
   c) What is the sum expected to be paid by the client?

**Exercise 107:**
A factory produces watches. Each watch is tested before being approved for selling. If the test is positive, that is if the watch functions properly, then the watch is approved for selling. But if the test is negative then the watch is repaired after which it is tested again. If its second test is positive then it is approved for selling, but if the test is negative then the watch is destroyed. It is known that:
for 80% of the watches produced, the first test is positive;
for 60% of the repaired watches, the second test is positive.
One watch is chosen randomly from the production.
1) Prove that the probability for this watch to be destroyed is 0.08.
2) Determine the probability that this watch is approved for selling.
3) The cost of production of a watch is 40 000LL with an additional cost of 10 000LL if it needs to be repaired. Each watch is sold for 70 000LL. Let X be the random variable that is equal to the profit achieved by the factory upon selling a watch.
   a) Verify that the three possible values of X are: –50 000, 20 000 and 30 000.
   b) Determine the probability distribution for X.
   c) Calculate the expected value E(X).
   d) Suppose that the daily production is 50 watches. Estimate the daily profit for this factory.
Exercise 108:

In order to encourage students to improve reading habits, a teacher uses two urns A and B such that
The urn A contains 6 white balls and 5 red balls.
The urn B contains 4 red balls and 7 green balls.
He proposes the following game:
The student draws at random one ball from the urn A:
If the drawn ball is white, then the student does not get anything.
If the ball is red, the student draws randomly a ball from urn B:
- If it is red, the student gets a gift of 10 books.
- If it is green, he again draws, without replacing the ball in B, another ball from B:
If this last ball is red, then he gets 5 books; if not, he does not get anything.
Consider the following events:
F: « The student gets 10 books ».
E: « The student gets 5 books ».
N: « The student does not get anything ».
1) What is the probability of the event: « the student does not get anything for the draw from urn A ».
2) Calculate the probability \( p(F) \) and show that \( P(E) = \frac{14}{121} \).
3) Calculate \( P(N) \).
4) Designate by \( X \) the random variable that is equal to the number of books received by the student. Find the expected value \( E(X) \).

Exercise 109:

A survey, carried out on the workers of a factory shows that: 30% of the workers speaks French, 25% speak English and 15% speak both languages. Consider the following events:

A: The chosen worker speaks French.
B: The chosen worker speaks English.
1) Calculate the probability of each of the following events:
a) \( A \cap B \)
b) \( A \cup B \)
c) The chosen worker speaks neither French nor English.
2) Calculate the following probabilities: \( P(A/B) \) & \( P(B/A) \).
3) Are the events A and B independent? Justify.
Exercise 110:
A certain store sells only jackets, coats and shirts.

During a week, 120 customers were served in this store.

90% of those customers bought each one jacket, while the other 30 customers bought each one coat.

40% of those who bought jackets bought each also a shirt, while 20% of those who bought coats bought each also a shirt.

A customer is chosen at random from those 120 customers and is interviewed.

1) Consider the following events:
\( J \) : « the interviewed customer has bought a jacket ».
\( C \) : « the interviewed customer has bought a coat ».
\( S \) : « the interviewed customer has bought a shirt ».

a) Verify that the probability of the event \( S \cap J \) is equal to \( \frac{3}{10} \).

b) Calculate the following probabilities:

\( P(S \cap C) \), \( P(S) \), \( P(C/S) \) and \( P(C/S) \).

2) The prices of the clothes in this store are as shown in the following table:

<table>
<thead>
<tr>
<th>Kind</th>
<th>Jacket</th>
<th>Coat</th>
<th>Shirt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price in LL</td>
<td>150 000</td>
<td>200 000</td>
<td>60 000</td>
</tr>
</tbody>
</table>

Let \( X \) designate the random variable that is equal to the amount paid by a customer.

a) Give the four possible values of \( X \).

b) Determine the probability distribution of \( X \).

c) Calculate the mean (expected value) \( E(X) \).

d) Estimate the amount of sales collected by the store during that week.
Simple interest-Compound interest-Annuity

Simple interest:
Calculation of simple interest:
The simple interest is computed by the formula $I = Prt$
P: The initial capital
r: The annual interest
t: The time in years

Acquired value (Future value):
$F = P + I = P(1 + rt)$

Actual value (present value):
$P = \frac{F}{1 + rt}$

Compound interest:
Future value (Acquired value):
The acquired (future) value after $n$ periods is: $F_n = P(1 + i)^n$
P: initial capital
R: the annual interest rate
t: time in years
i: the interest rate per period
n: the number of periods during t years where $i = \frac{r}{k}$ and $n = kt$

Present value (Actual value):
$P = F_n(1 + i)^{-n}$

<table>
<thead>
<tr>
<th>Interest compounded annually</th>
<th>$k = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest compounded semi annually</td>
<td>$k = 2$</td>
</tr>
<tr>
<td>Interest compounded quarterly</td>
<td>$k = 4$</td>
</tr>
<tr>
<td>Interest compounded monthly</td>
<td>$k = 12$</td>
</tr>
<tr>
<td>Interest compounded weekly</td>
<td>$k = 52$</td>
</tr>
<tr>
<td>Interest compounded daily</td>
<td>$k = 365$</td>
</tr>
</tbody>
</table>

Annuities:
R: the periodic payment
n: the number of periods
i: the interest rate per period

Future value (Acquired value): $F_n = R \frac{(1+i)^n-1}{i}$
The gained interest \( I = F_n - n \times R \)
Present value (Actual value): \( C_n = R \frac{1-(1+i)^{-n}}{i} \)
The paid interest: \( I = n \times R - C_n \)

**Simple interest-Compound interest-Annuity**

**Exercise 111:**
On the first of November 1996, a company offered a loan of $5000 at a simple interest rate of 10.4\%. Calculate the amount of money that should be paid to pay the loan on the first of November 2006.

**Exercise 112:**
A capital of $8400 produced from Mars 16 to September 25 of the same year, a simple interest of $231. Calculate the interest rate.

**Exercise 113:**
Calculate the capital placed at a simple interest rate of 8.4\% during 62 days, that acquired a value of $16738.7

**Exercise 114:**
Two capitals that differ by $1000 are placed at simple interest. The first capital, which is the larger, is placed at 12\% during 8 months; and the second at 10\% during 6 months. The interest produced by the first is the double of that produced by the second. Calculate these two capitals and the interest earned by each of them.

**Exercise 115:**
To buy a car, we borrowed $6500, at a simple interest rate 10\%, from January 3 until November 13 of the same year. Suppose that it is a leap year, what is the sum that we should pay at the maturity of the loan?

**Exercise 116:**
Calculate the rate of investment of a capital of $2500 that reported a total interest of $1375 in 5 years in each of the following cases.
1) The investment is made with a simple interest.
2) The investment is made with an annual compound interest.

**Exercise 117:**
We place $5000 for 5 years at an annual interest rate 4.5\% compound annually. What is the sum that we will earn in 5 years?

**Exercise 118:**
What is the sum of money that we should place today, at an annual interest rate of 5\% compounded annually, to earn $10000 in 4 years.

**Exercise 119:**
We place a sum of $5000 in a bank. In 3 years, we will earn $5921.14. What is the rate of interest if the interest is compounded annually?

**Exercise 120:**
We place two sums at a compound interest compounded annually: one of them is $10000 at 6% and the other is $9018.6 at 7%. In how many years will they acquire the same value? Calculate this value.

**Exercise 121:**
Two capitals with a sum of $10000 are placed in a bank, one of them at a simple interest rate of 10% and the other at composite interest rate 8% compounded annually. In 9 years, they will acquire the same value. Calculate these capitals.

**Exercise 122:**
Sami placed in a bank, for 6 years, a sum of 20 000000 L.L at an annual interest rate of 7% compounded quarterly.
His brother Walid placed in the same bank, for 6 years, an equal sum of 20 000000 L.L at an annual interest rate 8% compounded annually.
1) Calculate the capitals S and W acquired respectively by Sami and Walid during these 6 years.
2) Which of the two brothers made the more advantageous choice? Justify your answer.

**Exercise 123:**
A person placed a sum of $16000, at an annual interest rate of 12% compounded quarterly, for 3 years and 3 months. What is the amount of money that he will earn at the end of this period?

**Exercise 124:**
In how many years, a capital placed at an annual interest rate 10% compounded monthly, would triple?

**Exercise 125:**
Walid will register in the university on 1/10/2020 and he will have to pay 15000000 L.L. so that he can attend classes. What is the sum that he must place on 1/10/2010 at 8% compounded monthly which will allow Walid to continue his studies?

**Exercise 126:**
Sami wants to deposit 4000000 L.L for 6 years in a bank, which offers him two choices:
Choice S: A simple interest with an annual rate of 12%.
Choice T: A compound interest with an annual interest rate of 10% compounded quarterly.
1) Calculate the amount that Sami will earn if he chooses
   a) The choice S
   b) The choice T
2) Which of the two choices is more advantageous for Sami?

**Exercise 127:**
Calculate the acquired value of a series of 15 annuities each equals to 200000 L.L at an annual interest rate of 8.5%.

**Exercise 128:**
In order to constitute a capital of 100 000000 L.L, a man had to save money by paying constant annual payments for 10 years at an annual interest rate of 10%.
1) Calculate the amount of each payment.
2) What is the actual value of this series of annuities?

**Exercise 129:**
The price of a car is $20000.
1) Determine the number of months needed to pay back its cost with payments of $300 each at an annual interest 10% compounded monthly.
2) The sales man gives 5 years maximum to pay it back. Calculate in this case, the value of each payment.

**Exercise 130:**
To pay off a loan we have to pay a series of 12 annuities where each annuity is equal to 2000000 L.L at an annual interest rate 9.6% compounded monthly.

a) What is the amount of the loan?
b) What is the sum of money that we need to pay back the debt in one payment after four months?
c) We propose to the bank an offer to pay, in 12 months, a sum that is equal to 24 000000 L.L to pay off the loan. Does the bank accept this offer? Justify

**Exercise 131:**
To buy a car, Rami paid $4000 at the beginning then a sum of $400 each month for 3 years, at an annual interest rate of 12% compounded monthly.

1) Determine the price of the car.
2) Determine the interest paid by Rami.

**Exercise 132:**
The price of a car that Nabil desires to buy is 20000000 L.L. He borrowed a sum from the bank that proposed to him to pay back his loan by monthly payments during 5 years at an annual interest rate of 6% compounded monthly.

1) Calculate the amount of each payment.
2) What is the amount of interest paid by Nabil?

**Exercise 133:**
Three persons X, Y and Z each borrowed a sum of $30000. The contract of Mr. X consists of paying back the debt in one payment after 10 years at an annual interest rate of 6% compounded monthly. The contract of Mr. Y consists of paying back the debt by a series of 10 annuities each equal to “A” in $, paid at the end of each month at a rate of 8%. The contract of Mr. Z consists of paying back the debt by a series of 120 annuities each equal to “a” in $, paid at the end of each month at a rate of 9%.

1) Calculate the acquired value of the capital that Mr. X has to pay after 10 years.
2) Calculate the values of a and A.
3) Who, among the three persons, paid the highest interest?
Far a certain reason, Mr. Y didn’t pay the first three annuities. The bank forced him to pay back all of the debt in one payment at the end of the 4th annuity. Calculate the amount of money that Mr. Y has to pay.